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Jean-Paul Décamps and Stéphane Villeneuve

Toulouse School of Economics

Dynamics of cash holdings, learning about profitability, and access to the market*

Jean-Paul Décamps[†] Stéphane Villeneuve[‡]

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Abstract: We develop a dynamic model of a firm whose shareholders learn about its long-term profitability, face costs of external financing and costs of holding cash. Cash management policy generates a corporate life-cycle with two stages: a "probation stage" where the firm has no access to the capital markets, pays little in dividends, increases its cash target levels and a ''mature stage" where the firm does have access to external financing, pays dividends, decreases its cash target levels. The model provides new insights on the corporate propensity to save and the firm's value dynamics when its profitability is difficult to ascertain.

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[†]Toulouse School of Economics, University of Toulouse Capitole, TSM-R, Manufacture des Tabacs, 21, Allée de Brienne, 31000 Toulouse, France. Email: jean-paul.decamps@tse-fr.eu

[‡]Toulouse School of Economics, University of Toulouse Capitole, TSM-R, Manufacture des Tabacs, 21, Allée de Brienne, 31000 Toulouse, France. Email: stephane.villeneuve@tse-fr.eu

1 Introduction

In this paper, we explain how learning about long-term profitability impacts corporate cash management. While the trade-off between the costs and benefits of holding cash has received much attention in recent literature, little is known on the dynamics of this trade-off in a setting where long-term profitability is difficult to ascertain. Nevertheless, this issue is key for corporations that do not fully know their long-term prospects and face important external financing constraints. This is notably the case for young firms conducting intensive research and development (R&D) and innovative activities, as documented in the literature.[1] For those firms, information problems and lack of collateral value of intangible assets make external capital very costly. This leads them to finance their activities with internal cash flow and to issue stock when cash flow is exhausted, as pointed out in Brown, Fazzari and Petersen (2009).[2] Learning about profitability is key because better profitability prospects facilitate access to the market.[3]

These observations raise the more general issue of corporate cash policy in an incomplete information setting and echo the so-called corporate life-cycle theory, which develops the idea that the trade-off between the advantages and costs of cash retention evolve with profitability prospects. Corporate life-cycle theory has proven empirically relevant in explaining firms' payout practices and the dynamics and valuation of cash holdings.[4] In short, young firms tend to hold more cash and pay little in dividends because they are afraid of losing a potentially profitable project due to their limited access to external resources. Mature firms with proven profitability are more likely to pay dividends and have better access to capital markets. In this view, cash is built up to cover investment needs that the capital markets

[1] See, e.g., Hall and Lerner (2010) and Kerr and Nanda (2015) for surveys on the financing of innovation and the role of learning in running innovative projects.

[2] Brown, Fazzari and Petersen (2009) find evidence that young, high-tech firms almost entirely financed by cash flow or public share issues explain most of the 1994 to 2004 aggregate R&D cycle. More recently, Graham and Leary (2018) find evidence that a large number of new public Nasdaq firms from 1980 to 2000 were holding large amounts of cash. They find that this effect was most pronounced among unprofitable, largely debt-free, high-growth, and high-volatility firms operating in the health care or high-tech industries.

[3] A striking textbook case is the funding crisis Intel faced in the early 1980s, as documented in Passov (2003). At that time, the large potential of microprocessors was difficult to realize, which is a main reason external financing was extremely costly for Intel.

[4] See, for instance, DeAngelo, DeAngelo and Stulz (2006), who test the life-cycle theory of dividends, and DeAngelo, DeAngelo and Stulz (2010), who study the determinants of seasoned equity offerings and the relation to the corporate life cycle. See also Drobetz, Halling and Schroder (2015), who use the Dickinson (2011) methodology to relate the dynamics of corporate cash to the corporate life cycle.

would not be willing to finance and is less critical when the firm becomes mature.

The theoretical literature on corporate cash, despite the many achievements discussed below, does not capture these stylized facts. This is because, with few exceptions, existing models focus on liquidity issues in a complete information setting and typically remain silent on the intertwining between holding cash and learning about profitability. Our paper aims to close this gap. We develop a stylized continuous-time model of an all-equity firm confronted with three frictions: imperfect information about long-term profitability, external financing costs and costs of holding cash. Shareholders are cash constrained and do not know the actual long-term profitability of the firm's project. They observe at any time realized earnings and infer the profitability prospects from a Bayesian estimation of the firm's long-term profitability. They also control the dynamics of cash through issuance and payment policies, thereby weighing the costs and benefits of holding cash. In such a framework, shareholders must cope with both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of having to liquidate a profitable project). Formally, we solve a new two-dimensional optimization problem where the state variables are the controlled cash reserves and the profitability prospects. The problem is highly nontrivial. Intuitively, a positive shock to earnings increases the profitability prospects. This should facilitate external financing and induce the firm's management to lower cash target levels. Nevertheless, a firm has more to lose from liquidity constraints when profitability prospects are high than it does when they are low. This may induce the firm's management to accumulate more cash when the profitability prospects increase. We spell out all these interactions and show that they result in two life-cycle stages for the firm: a "probation stage" in which the firm has no access and a "mature stage" in which the firm does have access to capital markets. We provide a stylized theoretical model where the corporate life-cycle dynamics of cash holdings stem from the optimal equity issuance, payout and liquidation policies that we derive explicitly. A rich set of implications follows. We highlight our primary findings here.

Issuance occurs when cash reserves are depleted if and only if profitability prospects are larger than an endogenous threshold that corresponds to the minimal level of profitability prospects required to access the capital markets. Above that threshold, the firm is in the mature stage and can issue new shares whenever needed. Below that threshold, the firm is in the probation stage, cannot tap the market and is liquidated when cash is exhausted. Therefore, in our study, the terms "probation stage" and "mature stage" do not refer to the age of the firm but rather to its ability to finance its activity with new share issues. In particular, the firm can go back and forth between the two stages.

The uncertainty about the firm's actual profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its long-term profitability. We establish that a continuous function of the profitability prospects, *the dividend boundary function*, characterizes the cash target levels. We show that when cash reserves reach the dividend boundary function, shareholders pay out as dividends a fraction of the cash above the dividend boundary and reinvest the complement into the firm. The fraction that shareholders reinvest into the firm is also a function of the profitability prospects, the so-called *corporate propensity to save*. Our theoretical analysis yields dynamics of cash holdings that are drastically different in the two different regimes of the corporate life cycle. Salient results are as follows.

- In the probation stage, the precautionary motive for holding cash is strong because the firm has no access to the market. The model predicts that the dividend boundary function is increasing in the profitability prospects and that the corporate propensity to save is decreasing. The corporate propensity to save takes large positive values, which means that the firm pays little in dividends. The firm reaches its maximum cash target level on the edge between the two regimes.

- Shareholders of a firm entering into the mature stage have built a large amount of cash reserves and have increased profitability prospects. They optimally decide to initiate dividend payments. This causes a discontinuity in the corporate propensity to save, which becomes negative, meaning that the firm dissaves and uses its reserves to pay more dividends than its last profit. The corporate propensity to save is negative and increasing and tends toward 0 as profitability prospects increase. The dividend boundary function is decreasing in the profitability prospects and tends toward a cash target level that prevails in the complete information benchmark of our model.

- The profitability prospects required to enter the market, as well as the cash target levels, increase with the cost of external financing. As a result, the model predicts that a high-cost firm dissaves more aggressively when it becomes mature.

All these results are unique to our model and are grounded in a mathematical contribution to the literature on stochastic control: we analytically solve a two-dimensional Bayesian adaptive singular control problem. We prove that the associated free boundary function (our dividend boundary function) is defined as the solution to an ordinary differential equation. The free boundary function is nonmonotonic and attains its maximum at a point where it is nondifferentiable. We show that the corporate propensity to save is related to the derivative of the dividend boundary function. It follows that the corporate propensity to save features a discontinuity when the cash target level reaches its maximum, as we emphasized in the previous paragraph.

Additional economic insights follow from the model analysis. We find that a single indicator, defined as the firm performance, characterizes the two stages of the corporate life cycle. The firm's performance reflects payments to shareholders, the issuance proceeds and the ex ante prior beliefs on the firm's long-term profitability. The performance of the firm remains unchanged as long as the firm neither pays dividends nor issues new securities. It increases whenever the firm reaches a cash target level and decreases whenever the firm issues new shares. The initial assessment of the firm's long-term profitability (for example, by specialized intermediaries for the financing of innovation) has lasting effects and matters for evaluating the performance of the firm at any date. We find that the firm has access to the capital markets if and only if its performance is above an endogenous threshold.

We show that for a fixed level of performance, the firm value can be described as a function of its cash reserves. That function is increasing and convex for a fixed performance level sufficiently below the required performance to access the market, while it is increasing and concave for a fixed performance level sufficiently above the required performance to access the market. The value of the firm as a function of its cash reserves is neither convex nor concave for fixed levels of performance close to the market access threshold. These properties are unique to our model. Intuitively, in the probation stage, profitability is an important issue. One more unit of profitability prospects has a larger impact on the value of the firm than one more unit of cash. The opposite occurs in the mature stage. This yields the above concavity/convexity properties. A series of novel implications follow regarding the dynamics of the value of the firm, especially about the relationship between the volatility of the firm and its value. Our model predicts a positive relationship between the volatility of the firm and its value when the firm is far from accessing to the market and a negative relationship between firm value and volatility when the firm has proven access to the market. Close to the market access threshold, the volatility of the firm is an inverted U-shaped function of the value of the firm. Overall, our model suggests that most changes in the features of a firm's key indicators (volatility, cash target levels, payout ratios) occur at transition phases between life-cycle stages.

Relationship to the literature. Our paper belongs to the growing theoretical literature on corporate cash management. In the simplest models, the cumulative net cash flow generated by the firm follows an arithmetic Brownian motion. The constant drift represents the firm's profitability per unit of time, and the Brownian shock is interpreted as a liquidity shock. External financing is costly, which creates a precautionary demand for cash. Agency costs of free cash flow create a cost of carrying cash. This results in a unique optimal payout policy that requires paying shareholders 100% of earnings beyond an endogenous constant cash target level (the corporate propensity to save at the cash target level is zero). Pioneering studies include Jeanblanc and Shiryaev (1995), Radner and Shepp (1996) and Kim, Mauer and Sherman (1998).[5] These contributions have been extended in a number of directions. For instance, Décamps, Mariotti, Rochet and Villeneuve (DMRV) (2011) study the interaction between cash management, agency costs, issuance costs and stock price; Bolton, Cheng and Wang (2011) extend the model to the case of flexible firm size in order to study the dynamic patterns of corporate investment; and Bolton, Chen and Wang (2013) and Hugonnier, Malamud and Morellec (2014) introduce capital supply uncertainty and the necessary time needed to secure outside funds into the analysis. Décamps, Gryglewicz, Morellec and Villeneuve (2017) assume that the firm's operating cash flow is proportional to profitability, the dynamics of which are governed by a geometric Brownian motion. The ratio of cash holdings to profitability is the state variable of the firm's problem. This leads to a dividend boundary function that is linear in profitability. Malamud and Zucchi (2018) develop a model of endogenous growth where firms face costly access to financial markets and hoard cash reserves to sustain innovation. Closer to our study, Bolton, Wang and Yang (BWY) (2019) develop a two-dimensional real option model with financial constraints. The state variables are the earnings which follow a geometric Brownian motion process, and the controlled cash reserve process. Investment decisions in growth options characterize the different phase of the firm's life-cycle. BWY (2019) study the interaction between optimal timing of real options and corporate cash management. We study the interaction between learning about profitability and corporate cash management.

Our study is naturally related to the corporate finance literature that emphasizes the role of learning about profitability and its importance for corporate decision-making. Pastor and Veronesi (2003) study stock prices in a model in which shareholders of an all-equity firm learn about profitability over time. Their model avoids both the liquidity and liquidation issues, assuming a peculiar dividend strategy that maintains a positive book value of the firm's equity at any time. In Moyen and Platinakov (2013), shareholders update their beliefs

[5]Influential empirical papers driving the theory are Opler, Pinkowitz and Stulz (1999) and Bates, Kahle and Stulz (2009).

about firms' quality (high or low) in a dynamic Tobin's q framework. They find evidence that firms with unclear quality are more sensitive to earnings in their investment decisions than are well-established firms. In these models, cash-constrained firms become well established as time passes by learning about their profitability. Unlike in our model, cash management plays no role in these models. DeMarzo and Sannikov (2017) study a dynamic contracting model with learning about the profitability of the firm. In their model, asymmetric information arises endogenously because by shirking, an entrepreneur can distort the beliefs of investors about the project's profitability. The paper studies the relationship between incentives and learning. Our focus is different. We do not model hidden actions. In our model, information is incomplete but symmetric between stakeholders. We study the interplay between the evolution of profitability prospects and the evolution of the trade-off between the cost and benefit of holding cash. Our learning technology is also different. We borrow it from Decamps, Mariotti and Villeneuve (2005), who study the optimal decision to invest in a project whose profitability is not perfectly known. There are no financial constraints in their setting, so payout policies and issuance policies are irrelevant. Our paper is closely related to Gryglewicz (2011), who also considers a model in which the profitability per unit of time is a random variable, the value of which shareholders learn over time. There are no frictions inside the firm, so holding cash is not costly. It follows that any dividend policy that maintains the reserves above a critical level is optimal. Gryglewicz (2011) studies how this framework impacts the optimal capital structure that results from the trade-off between tax shields and bankruptcy costs. In our study, holding cash and issuing new shares are costly.[6] The payout and issuance policies are uniquely defined.

The paper is organized as follows. We lay out the model in Section 2. Section 3 studies benchmarks in which shareholders face profitability and liquidity concerns separately. Section 4 solves the model in a closed form and presents the optimal corporate policies. Section 5 develops the model implications. Section 6 concludes. All the proofs are in the Appendix.

2 The model

2.1 Learning

A firm has a single investment project that generates random cash flows over time. The cumulative cash flow process $\{R_t; t \geq 0\}$ follows an arithmetic Brownian motion with unknown

[6]In Gryglewicz (2011), equity financing is costless beyond $t = 0$, as is often the case in contingent claim models.

drift Y and known variance σ

$$dR_t = Y\,dt + \sigma\,dW_t, \quad t \geq 0,$$

where $\{W_t; t \geq 0\}$ is a standard Brownian motion independent of Y.[7]

The firm is held by risk-neutral shareholders who do not fully know the project's long-term profitability Y but observe the cumulative cash flow process $\{R_t; t \geq 0\}$. The project's profitability Y takes either of the two values $-\mu < 0 < \mu$. The conditional expectation

$$Y_t = \mathbb{E}[Y \mid \mathcal{F}_t^R], \tag{1}$$

defines the profitability prospects at time t. In (1), $\{\mathcal{F}_t^R; t \geq 0\}$ denotes the filtration generated by $\{R_t; t \geq 0\}$ that models the flow of information available to shareholders. The process $\{Y_t; t \geq 0\}$ satisfies the filtering equation[8]

$$dY_t = \frac{1}{\sigma}(\mu^2 - Y_t^2)dB_t, \tag{2}$$

where the so-called innovation process $\{B_t; t \geq 0\}$ is a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t^R; t \geq 0\}$ and satisfies

$$dB_t = \frac{1}{\sigma}(dR_t - Y_t dt). \tag{3}$$

The cumulative cash flow process is a sufficient statistic for Bayesian updating. Specifically, a direct application of Itô's formula yields the relation

$$dR_t = d\phi(Y_t), \tag{4}$$

where the function $\phi(y) = \dfrac{\sigma^2}{2\mu} \ln\left(\dfrac{\mu + y}{\mu - y}\right)$ is increasing and defined on $(-\mu, \mu)$. Finally, we obtain from (3) that

$$\mathbb{E}\left[\int_0^\infty e^{-rs}\,dR_s\right] = \mathbb{E}\left[\int_0^\infty e^{-rs} Y_s\,ds\right] \leq \frac{\mu}{r}.$$

Therefore, the present value of the future cash flows is lower than the present value of future cash flows of a project with observed long-term profitability μ.

[7]Technically, the process $\{W_t; t \geq 0\}$ and the random variable Y are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with filtration $\{\mathcal{F}_t; t \geq 0\}$ satisfying the usual condition of right continuity and completion by $\mathbb{P}$-negligible sets.

[8]See, for instance, Liptser and Shiryaev (1977).

2.2 The shareholders' problem

Risk-neutral shareholders discount future payments at the risk-free interest rate $r > 0$ and must keep positive liquid reserves at all times if they want to avoid liquidation. The model builds on the standard *cost versus benefit trade-off of holding cash*. The firm accumulates cash for precautionary motives in a costly external financing environment. We allow the firm to increase its cash holdings or cover operating losses by raising funds in the capital markets. External financing involves a proportional cost $p > 1$: for each dollar of new shares issued, the firm only receives $1/p$ dollars in cash, where $p - 1 > 0$ represents the marginal issuance cost. Carrying cash is costly. Because of internal frictions, the rate of return of cash inside the firm is lower than the cost of capital. To capture in a simple and tractable way the agency costs of free cash flow, we consider that the cost per unit of time of holding one dollar of cash inside the firm corresponds to the cost of capital $r\,dt$. Shareholders can reduce these costs by deciding to distribute cash.[9] In addition to this trade-off, shareholders are not aware of the profitability, positive or negative, of the firm's project. Thus, shareholders face both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of having to liquidate a profitable project).

Formally, at each date, shareholders decide whether to continue the project, whether to distribute dividends and whether to issue new shares. For simplicity, we assume that the liquidation value of the project is equal to 0. The cumulative cash reserve process $X = \{X_t; t \geq 0\}$ evolves according to the dynamics

$$dX_t = dR_t + \frac{dI_t}{p} - dD_t, \tag{5}$$

where $D = \{D_t; t \geq 0\}$ and $I = \{I_t; t \geq 0\}$ are nondecreasing processes that denote the cumulative dividend and the cumulative issuance processes.[10] Thus, dX_t, the cash reserves at time t, corresponds to the operating cash flow dR_t *plus* the cash flow from financing activities $\frac{dI_t}{p} - dD_t$, that is, the cash received from issuing securities minus the cash paid as dividends.

[9] These modeling assumptions are standard in corporate cash models. Agency costs of free cash flow can be notably important in the innovation sector due, for instance, to moral hazard between the inventor and financiers. See, e.g., Allen and Michaely (2003), DeAngelo, DeAngelo and Skinner (2008), and Hall and Lerner (2010) for insightful surveys of the literature.

[10] Technically, (I, D) belongs to the set $\mathcal{A}$ of admissible policies: I and D are $\mathcal{F}^R$-adapted, right-continuous, and nondecreasing with $I_0 = D_0 = 0$, and the associated cash reserves $\{X_t, \mathcal{F}_t^R; t \geq 0\}$ satisfy $X_t \geq 0$, $e^{-rt}X_t$ integrable and, $\lim_{t\to\infty} \mathbb{E}[e^{-rt}X_t] = 0$. Note that D is nondecreasing, so shareholders' limited liability is satisfied. In our setting, dividends can be distributed at no cost, and we do not need to allow share repurchases; thus, I is nondecreasing.

Using (4), we rewrite (5) in the form

$$X_t = \phi(Y_t) - \phi(Y_0) + X_0 + \frac{I_t}{p} - D_t. \tag{6}$$

Equation (6) is an accounting identity that specifies the relationships between cash reserves, profitability prospects, cumulative issuance and cumulative dividends. Observe that since ϕ is an increasing function, by holding the cash reserves X_t fixed and the cumulative issuance I_t fixed, the higher the cumulative dividend D_t is, the larger the profitability prospects Y_t. Thus, Equation (5) shows a positive relationship between cumulative dividend and profitability prospects, all else being equal. Similarly, there is a negative relationship between the cumulative issuance and the profitability prospects, all else being equal. Additionally, by holding the cumulative dividend and the cumulative issuance fixed, there is a one-to-one relationship between the cash reserves and the profitability prospects.

The firm ceases its activity for two possible reasons: (i) the firm cannot meet its short-term operating costs by issuing new shares or by drawing cash from its reserves, and/or (ii) shareholders strategically decide to liquidate because the profitability prospects are not high enough. Thus, equation (5) represents the dynamics of the cash reserves up to the liquidation time τ_0 defined as

$$\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}.$$

Given an admissible dividend policy $(I, D) \in \mathcal{A}$, current cash reserves $x \in [0, \infty)$ and current profitability prospects $y \in (-\mu, \mu)$, the value of the firm corresponds to the expected present value of all future dividends minus the expected present value of all future gross issuance proceeds

$$V(x, y; I, D) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dD_t - dI_t)\right], \tag{7}$$

with $D_{\tau_0} - D_{\tau_0^-} = \max(X_{\tau_0^-}, 0)$. The case $D_{\tau_0} - D_{\tau_0^-} = X_{\tau_0^-} > 0$ corresponds to a strategic liquidation.[11] The shareholders' problem is to find the optimal value function defined as the supremum of (7) over all admissible issuance and dividend policies

$$V^*(x, y) = \sup_{(I,D)\in\mathcal{A}} V(x, y; I, D). \tag{8}$$

3 Benchmarks

Two polar cases are worth mentioning: (i) the case where shareholders face only a profitability concern and (ii) the case where the shareholders face only a liquidity concern. These

[11] In that case, shareholders take the remaining cash reserves $X_{\tau_0^-} > 0$ as dividends.

benchmarks provide natural bounds to the value of the firm V^*.

3.1 First-best benchmark

Let us assume that shareholders can issue and repurchase shares at no cost whenever they want. That is, we assume that $p = 1$ in (5). In this framework, shareholders face only a profitability concern. Accumulating cash does not bring any benefit, while carrying cash is costly. Intuitively, an optimal policy is to distribute all of the firm's initial cash reserves x as a special payment at date 0, to hold no cash beyond that time, and to liquidate when the profitability prospects are too low. In this first-best environment, the firm's value $\hat{V}$ is equal to the sum of current cash reserves plus the option value to liquidate the firm[12]:

$$\hat{V}(x, y) \equiv x + \sup_{\tau \in \mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs} Y_s ds\right]. \tag{9}$$

The firm is optimally liquidated when the firm's profitability prospects hit the so-called first best liquidation threshold y^*, which can be explicitly computed. Formally, the stopping time $\tau^* = \inf\{t \geq 0 \,|\, Y_t = y^*\}$ is optimal for problem (9). The mapping $y \longrightarrow \hat{V}(x, y) - x$ is an increasing and convex function over $(-\mu, \mu)$. It corresponds to the option value to liquidate the firm.

The logic of Miller and Modigliani (1961) applies, and there is a large degree of freedom in designing the cumulative issuance and dividend processes that deliver the value (9). To see this, consider the payout policy $\hat{D}_t = (x\mathbb{1}_{t=0} + lt)\mathbb{1}_{t \leq \tau^*}$. That is, the firm distributes all its cash reserves at time 0 and pays a constant dividend flow $l > 0$ up to the stopping time τ^*, which corresponds to the date at which shareholders strategically decide to liquidate. To keep cash reserves constant and equal to zero after time 0, shareholders issue or repurchase new shares to offset earnings up to the liquidation date τ^*, that is, $\hat{I}_t = ((Y_t - l)dt + \sigma dB_t)\mathbb{1}_{t \leq \tau^*}$. Thus, we have

$$\hat{V}(x, y) = \mathbb{E}\left[\int_0^{\tau^*} e^{-rt}(d\hat{D}_t - d\hat{I}_t)\right] = x + \sup_{\tau \in \mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs} Y_s ds\right].$$

To summarize,

Proposition 1 *Suppose that shareholders face only a profitability concern. Then, the following holds:*

(i) The value of the firm, $\hat{V}(x, y)$, is an increasing and convex function of the level y of the profitability prospects.

[12] In (9), we denote by $\mathcal{T}^R$ the set of $\mathcal{F}^R$-stopping times.

(ii) Distributing all initial cash reserves x at time 0, holding no cash beyond that time, and liquidating at the stopping time $\tau^* = \inf\{t \geq 0 \;:\; Y_t = y^*\}$ are an optimal policy. There are many degrees of freedom in designing the dividend policy.

Finally, the shareholders' value function V^* in (8) is bounded above by the benchmark $\hat{V}$.

Corollary 1 *The value functions V^* and $\hat{V}$ satisfy $V^*(x,y) \leq \hat{V}(x,y)$ for any $(x,y) \in [0,\infty) \times (-\mu,\mu)$.*

In particular, if the profitability prospects are lower than the first-best liquidation threshold y^*, then the firm is liquidated regardless of the amount of cash within the firm. Formally, $V^*(x,y) = \hat{V}(x,y) = x$ for all $x \geq 0$ and $y \leq y^*$.

3.2 Complete information benchmark

Another useful benchmark is the complete information setting in which shareholders face only a liquidity concern. This corresponds to the case where $y = -\mu$ or $y = \mu$ in the main problem (8).[13] We shall denote by $V_{-\mu}(x)$ and $V_\mu(x)$ the associated values of the firm.

When $y = -\mu$, the firm's profitability is negative, and it is optimal for shareholders to take the initial cash reserves and to liquidate the firm at time $t = 0$. We have that $V_{-\mu}(x) = x$, $\forall x \geq 0$.

When $y = \mu$, the dynamics of the cash reserve process take the form

$$dX_t \;=\; \mu\, dt + \sigma\, dW_t \,+\, \frac{dI_t}{p} \,-\, dD_t.$$

and we revert to a classic case studied by many authors.[14]

Specifically, the value of the firm, $V_\mu(x)$, is an increasing and concave function of the level x of its cash reserves. The marginal value of cash, $V_\mu'(x)$, is strictly greater than one up to $x_\mu = \inf\{x > 0 \,|\, V_\mu'(x) = 1\}$. The threshold x_μ corresponds to the firm's cash target level at which dividends are paid. If cash holdings x exceed x_μ, the firm places no premium on internal funds, and it is optimal to make a lump sum payment $x - x_\mu$ to shareholders. Accordingly, $V_\mu(x) = x - x_\mu + V_\mu(x_\mu)$ for any $x \geq x_\mu$.

Because external financing is costly, it is optimal to postpone the issuance of new shares for as long as possible: equity issuance only takes place whenever cash reserves are depleted

[13] Technically, $-\mu$ and μ are absorbing barriers for the process Y: if $y = \mu$ (resp. $y = -\mu$), then $Y_t = \mu$ a.s (resp. $Y_t = -\mu$ a.s).

[14] See, for instance, the textbook by Moreno-Bromberg and Rochet (2018).

and occurs if and only if the cost of issuance is not too high. Specifically, there exists a threshold $\overline{p}$ such that there is equity issuance every time $X_t = 0$ if and only if $p < \overline{p}$.[15] In that case, the marginal benefit, $V_\mu'(0)$, is equal to the proportional issuance cost, p. Given that the value of the firm is concave in x, one obtains $V_\mu'(x) \leq p$ for $x \geq 0$. This means that it is indeed never optimal to issue new shares before cash reserves are depleted. Finally, the optimal issuance strategy induces a reflection at level zero of the cash reserve process so that infinitesimal amounts of new equity are issued every time $X_t = 0$.

The following proposition summarizes these standard results, first established in Lokka and Zervos (2008) and then used and generalized in several studies, especially in DMRV (2011) and Bolton, Chen and Wang (2011). We provide a complete and rigorous statement in appendix A.

Proposition 2 *Suppose that shareholders face only a liquidity concern, so $y = \mu$. Then, the value of the firm, $V_\mu(x)$, is an increasing and concave function of the level x of its cash reserves. Any excess of cash over the dividend boundary x_μ is paid out to shareholders. Furthermore,*

(i) If issuance costs are high such that $p \geq \overline{p}$, then it is never optimal to issue new equities, and the firm is liquidated as soon as it runs out of cash.

(ii) If issuance costs are low such that $p < \overline{p}$, then equity issuance takes place whenever the firm runs out of cash, so that the cash reserve process is reflected back whenever it hits 0, and the firm is never liquidated.

The complete information case also provides a useful upper-bound to the shareholders' value function V^*.

Corollary 2 *The value functions V^* and V_μ satisfy $V^*(x,y) \leq V_\mu(x)$ for any $(x,y) \in [0,\infty) \times (-\mu,\mu)$.*

The properties described in Proposition 2 are common to most of the recent models on corporate cash management: the value of the firm decreases after dividend payments and at dividend payment dates takes the constant value $V_\mu(x_\mu)$. At the cash target level, the firm has no propensity to save cash whatever the level of the financing frictions. That is, 100% of cash available for distribution is paid to shareholders. The value of the firm increases after

[15] The analysis yields that $\overline{p} = \overline{V}_\mu'(0)$, where $\overline{V}_\mu$ corresponds to the value of the firm if issuance of new shares is not allowed (see Appendix A).

issuances and takes at issuance dates the constant value $V_\mu(0)$. Additionally, in this theory, there are only two types of firms: firms that never default and firms that always default when cash reserves are depleted. We show that imperfect information about long-term profitability dramatically impacts all these results.

4 Model solution

The next section is heuristic and leads to a variational system that should satisfy the value of the firm V^* solution to (8).

4.1 Heuristic discussion

Taking the cash reserve and liquidating is an admissible policy, thus the value function V^* satisfies the inequality $V^*(x,y) \geq x$ for all $(x,y) \in (0,\infty) \times (-\mu,\mu)$. From Corollary 1, $V^*(x,y) = x$ for all $y \leq y^*$, where y^* is the first-best liquidation threshold. To proceed further, we assume in this section that V^* is as smooth as necessary, and we derive some properties that V^* should satisfy.[16]

Dynamic programming. Let us fix some pair $(x,y) \in (0,\infty) \times (-\mu,\mu)$. Let us consider the policy that consists of abstaining from issuing new shares and paying dividends for $t \wedge \tau_0$ units of time and, then, in applying the optimal policy associated with the resulting couple $(x + \int_0^{t\wedge\tau_0} Y_s\, ds + \sigma dB_s,\ y + \int_0^{t\wedge\tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2) dB_s)$, implied by dynamics (2) and (3). This policy must yield no more than the optimal policy:

$$\begin{aligned} 0 \ &\geq\ \mathbb{E}\left[e^{-r(t\wedge\tau_0)} V^*\left(x + \int_0^{t\wedge\tau_0} Y_s\, ds + \sigma dB_s, y + \int_0^{t\wedge\tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2) dB_s\right)\right] \\ &\quad -V^*(x,y) \\ &=\ \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\left(\mathcal{L}V^*(X_s,Y_s) - rV^*(X_s,Y_s)\right) ds\right]. \end{aligned} \tag{10}$$

The last equality follows from Itô's formula, where $\mathcal{L}$ denotes the partial differential operator defined by

$$\mathcal{L}V^*(x,y) = \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V^*_{yy} + \frac{1}{2}\sigma^2 V^*_{xx} + (\mu^2 - y^2) V^*_{xy} + y V^*_x.$$

Letting t go to zero in (10) yields

$$\mathcal{L}V^*(x,y) - rV^*(x,y) \leq 0$$

[16] We prove in the appendix all the regularity properties required to justify our model.

for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$.

Dividend boundary. The intuition that underlies the complete information benchmark applies: fix some $(x, y) \in (0, \infty) \times (-\mu, \mu)$; the policy that consists of making a payment $\varepsilon \in (0, x)$, and then immediately executing the optimal policy associated with cash reserves $x - \varepsilon$ must yield no more than the optimal policy. That is,

$$V^*(x, y) \geq V^*(x - \varepsilon, y) + \varepsilon.$$

Subtracting $V^*(x-\varepsilon, y)$ from both sides of this inequality, dividing through by ε and letting ε approach 0 yield

$$V_x^*(x, y) \geq 1$$

for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$. It is expected that the inequality $V_x^*(x, y) > 1$ holds for any $x \in (0, b^*(y))$, where $b^*(y) = \inf\{x,\, V_x^*(x, y) = 1\} > 0$. Intuitively, for any fixed profitability prospects y, any excess of cash above $b^*(y)$ should be paid out. Therefore, the optimal cash policy should not be characterized by a constant threshold, as in the previous literature, but rather by a dividend boundary function $y \longrightarrow b^*(y)$.

Issuance policy. If it is never optimal to issue new shares when the firm's profitability is known and equal to μ, then it should also never be optimal to issue new shares in the incomplete information setting. Thus, if $p \geq \overline{p}$, we expect that there are no equity issuances at all, so the firm is liquidated when it runs out of cash. If $1 < p < \overline{p}$, the logic of the complete information benchmark applies again: if there is any issuance activity, this must be when cash reserves drop down to zero to avoid liquidation. In such a situation, the marginal value of cash should be equal to the proportional issuance cost p, formally, $V_x^*(0, y) = p$. Intuitively, this latter equality should require that the profitability prospects when the cash reserves are depleted are sufficiently high. Accordingly, we conjecture the existence of an (endogenous) threshold y_i^* such that $V_x^*(0, y) = p$ for any $y \geq y_i^*$, whereas $V^*(0, y) = 0$ for any $y \leq y_i^*$. In this latter case, the profitability prospects are too low with regard to the cost of external financing, and the firm defaults when the cash reserves are depleted.

Convergence toward the complete information benchmark. Finally, we expect that when shareholders are increasingly confident that the profitability of the firm is μ, the value of the firm tends to the one derived in the complete information benchmark. We should have for all $x \geq 0$

$$\lim_{y \longrightarrow \mu} V^*(x, y) = V_\mu(x).$$

Our heuristic discussion leads us to consider the variational system: find a smooth function V, a constant $y_i \in (-\mu, \mu)$ and a positive function b continuously differentiable almost everywhere over $(-\mu,\mu)$ that solve

$$\begin{aligned}
\mathcal{L}V(x,y) - rV(x,y) &= 0 \quad \text{on the domain } \{(x,y),\ 0<x<b(y),\ -\mu<y<\mu\}, && (11)\\
V(0,y) &= 0 \quad \forall y \in (-\mu, y_i], && (12)\\
V_x(0,y) &= p \quad \forall y \in [y_i, \mu), && (13)\\
V_x(x,y) &= 1, \text{ for } x \geq b(y), && (14)\\
V_{xy}(b(y),y) &= 0, && (15)\\
\lim_{y\longrightarrow\mu} V(x,y) &= V_\mu(x) \ \ \forall x \geq 0. && (16)
\end{aligned}$$

Equation (15) ensures that the mapping $x \longrightarrow V(x,y)$ is twice continuously differentiable at the boundary function b so that the so-called super contact condition holds in our two-dimensional model.[17]

We prove the existence and the uniqueness of a solution (V, y_i^*, b^*) to the system (11)-(16). We show that the function V coincides with the value of the firm V^*. We obtain an analytical formula for the triple (V, y_i^*, b^*).

4.2 Solution to the shareholders' problem

In this section, we focus on the case $1 < p < \overline{p}$ and explain how to solve the system (11)-(16).[18] Then, we state our main result. Our analysis relies on a simple change of variable, which will be proved very useful for both the mathematical treatment and the economic analysis of the model.

We remark from equation (6) that, as long as the controls I and D are not activated, the process $Z = \{Z_t; t \geq 0\}$ with

$$Z_t \equiv \phi(Y_t) - X_t \tag{17}$$

remains constant. That is, keeping unchanged issuance and dividend policies, there is a one-to-one mapping between the cash reserves and the profitability prospects. The change of variable (17) allows us to restate problem (11)-(16) in the (z, y)-space and solve it analytically. The change of variable (17) also provides new economic insights that we comment on below.

[17]Specifically, differentiating equation $V_x(b(y), y) = 1$ with respect to y and using $V_{xy}(b(y), y) = 0$ lead to $V_{xx}(b(y), y) = 0$. We refer to Dumas (1991) for an insightful discussion of the super contact condition as an optimality condition for singular control problems.

[18]We refer the reader to the appendix for the complete analysis.

To develop the intuition, let us consider some admissible issuance and dividend policies I and D leading to cash reserve process

$$X_t = \phi(Y_t) - \phi(Y_0) + X_0 - D_t + \frac{I_t}{p}.$$

It follows that

$$Z_t = \phi(Y_t) - X_t = D_t - \frac{I_t}{p} + (\phi(Y_0) - X_0).$$

The process Z_t corresponds to the cash outflows from financing activities, $D_t - \frac{I_t}{p}$, corrected for the initial amount $\phi(Y_0) - X_0$. The process Z_t increases whenever the firm reaches a cash target level and decreases whenever the firm issues new shares. It measures the performance record of the firm at time t and defines a one-to-one mapping between profitability prospects and cash reserves that holds true as long as the firm neither pays dividends nor issues new securities.

Therefore, in our two-dimensional setting, the determinants of the shareholders' policy at a given date t are the cash position X_t and the profitability prospects Y_t; or, equivalently, the firm's performance Z_t and the profitability prospects Y_t; or, also equivalently, the cash position X_t and the firm's performance Z_t. We will use these three points of view to develop the economic implications of our model.

It is worth noting that the current cash outflow from financing activities, $D_t - \dfrac{I_t}{p}$, is not a sufficient statistic to define the performance of the firm at date t. The firm's performance at date t also depends on the initial profitability prospects Y_0 through the relation $\phi(Y_0) - X_0$. The initial profitability prospects Y_0 are not directly observable and follow, for instance, from a specific analysis by financial analysts of the relevance of the firm's project at the early stage of the firm's life. Thus, the performance of the firm is defined in light of the initial assessment of the profitability prospects. We will see that the firm's performance process Z indicates whether the firm can issue new shares if needed.

Using the change of variable (17), we define

$$U(z, y) \equiv V(\phi(y) - z, y),$$

and we restate problem (11)-(16) in the (z, y)-space.

In the (z, y)-space, the equation

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy} + \frac{1}{2}\sigma^2 V_{xx} + (\mu^2 - y^2)V_{xy} + yV_x - rV(x, y) = 0$$

becomes

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z,y) - rU(z,y) = 0. \tag{18}$$

The solution to (18) is explicit and can be written in the form

$$U(z,y) = A(z)h_1(y) + B(z)h_2(y), \tag{19}$$

where $h_1(y) = (\mu + y)^\gamma(\mu - y)^{1-\gamma}$ and $h_2(y) = (y + \mu)^{1-\gamma}(\mu - y)^\gamma$, with γ being the negative root of the equation $x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0$. The functions A and B are to be determined.

The condition $V(0,y) = 0$ for all $y \leq y_i$ becomes

$$U(\phi(y), y) = U(z, \psi(z)) = 0, \text{ for all } z \leq z_i,$$

where $z_i \equiv \phi(y_i)$ and where $\psi(z) \equiv \phi^{-1}(z) = \mu\frac{e^{\frac{2\mu}{\sigma^2}z} - 1}{e^{\frac{2\mu}{\sigma^2}z} + 1}$. Thus, for a given performance z, the real number $\psi(z) \in (-\mu, +\mu)$ corresponds to the profitability prospects when cash reserves are depleted, that is, when $x = 0$.

The condition $V_x(0,y) = p$ becomes

$$U_z(z, \psi(z)) = -p,$$

which holds for any $z \geq z_i$. In other words, the firm issues new shares when cash reserves are depleted only if its performance is higher than $z_i \equiv \phi(y_i)$ (equivalently, if the profitability prospects when the cash reserves are depleted are higher than y_i).

The condition $V_x(b(y), y) = 1$ becomes

$$U_z((\phi - b)(y), y) = U_z(z, k(z)) = -1,$$

where $k(z) \equiv \inf\{y \,|\, (\phi - b)(y) = z\}$. Thus, according to the change of variable (17), $k(z)$ corresponds to the profitability prospects at the cash target level $b(k(z)) = \phi(k(z)) - z$. Therefore, in our two-dimensional setting, each performance z defines a cash target level $b(k(z))$, the set of which forms the dividend boundary function. We will see that the function $\phi - b$ is invertible, so $k(z) = (\phi - b)^{-1}(z)$. We will prove that k is an increasing function. Intuitively, the higher the firm's performance is, the higher the profitability prospects at the cash target level.

The super contact condition $V_{xy}(b(y), y) = 0$ becomes

$$U_{zy}((\phi - b)(y), y) = U_{zy}(z, k(z)) = 0.$$

The convergence condition to the complete information benchmark, $\lim_{y \longrightarrow \mu} V(x, y) = V_\mu(x)$, becomes

$$\lim_{z \longrightarrow \infty} U(z, \psi(x + z)) = V_\mu(x).$$

Indeed, the change of variable (17) leads to $y = \psi(x + z)$ and, in turn, $V(x, \psi(x + z)) = U(z, \psi(x + z))$.

Overall, the free boundary problem (11)-(16) writes in the (z, y)-space in the following form: find a function U, a constant z_i, and a function k that solve the variational system

$$\begin{aligned} \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) &= 0 \quad \text{on the domain} \\ &\qquad \{(z, y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}, && (20) \\ U(z, \psi(z)) &= 0 \quad \forall z \leq z_i, && (21) \\ U_z(z, \psi(z)) &= -p \quad \forall z \geq z_i, && (22) \\ U_z(z, y) &= -1, \text{ for } k(z) \leq y, && (23) \\ U_{xy}(z, k(z)) &= 0. && (24) \\ \lim_{z \longrightarrow \infty} U(z, \psi(x + z)) &= V_\mu(x). && (25) \end{aligned}$$

We obtain an analytical solution (U, z_i^*, k^*) to the system (20)-(25) and thus a solution (V, y_i^*, b^*) to the system (11)-(16) through the relations $U(z, y) = V(\phi(y) - z, y)$, $k(z) = (\phi - b^*)^{-1}(z)$ and, $z_i^* = \phi(y_i^*)$. Below, we explain informally how we solve the system (20)-(25) and state our main result.

We consider a solution (U, z_i, k) to the system (20), (21), (22), (23), (24). For $z \geq z_i$, we obtain from (19), (22), (23), (24) that the function k is implicitly defined by the equation

$$-h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z)) + \frac{2\mu^2}{y^*}p = 0. \qquad (26)$$

We show that (26) defines over $(z_i, +\infty)$ a unique continuously differentiable increasing function k. Then, we set $y^i \equiv k(z_i)$, and we deduce from (19), (21), (23), (24) that, for $z \leq z_i$, the function k satisfies an ordinary differential equation, the terminal condition of which is $k(z_i) = y^i$. Finally, we show that there exists a unique $z_i = z_i^*$ such that (25) is satisfied. In turn, this uniquely determines the solution (U, z_i^*, k^*) to the system (20)-(25).

Returning to the formulation of the problem in the (x, y)-space, we characterize the dividend boundary function b^*. A main result is that the dividend boundary function b^* is nonmonotonic in the profitability prospects y. It reaches its maximum at $y^{i*} \equiv k(z_i^*)$. The threshold y^{i*} corresponds to the profitability prospects at the cash target level associated with the performance $z_i^* = \phi(y_i^*)$.[19] When the issuance cost p satisfies $p \geq \overline{p}$, we show that the firm never issues new shares and that the dividend boundary function b^* is increasing in the profitability prospects. The next theorem summarizes and completes these findings.

Theorem 1 *The value of the firm V^* solving (8) coincides with the unique solution (V, y_i^*, b^*) to the system (11)-(16). The threshold y_i^* corresponds to the required profitability prospects above which the firm issues new shares when the cash reserves are depleted and below which the firm is liquidated when it runs out of cash. Payments are made whenever cash reserves hit the dividend boundary function b^*. The function b^* is continuous over $[-\mu, \mu]$ and satisfies $b^*(y) = 0$ for $y \leq y^{**}$ and $b^*(\mu) = x_\mu$, where x_μ is the constant dividend boundary of the complete information benchmark. The threshold $y^{**} = max\{y \in (-\mu, \mu) \,|\, y = b^{*-1}(0)\} > y^*$ is well defined and corresponds to the minimum profitability prospects required by shareholders to run the project. The optimal cash reserve process is reflected along the function b^* in a horizontal direction on the (x, y)-plane. Furthermore:*

If the proportional issuance cost p satisfies $p \geq \overline{p}$,

- *$y_i^* = \mu$, so it is never optimal to recapitalize the firm. The firm is liquidated when it runs out of cash, $V^*(0, y) = 0$ for all $y \in (-\mu, \mu)$.*
- *The dividend boundary function b is continuously increasing and differentiable.*

If the proportional issuance cost p satisfies $1 < p < \overline{p}$,

- *$y_i^* \in (y^{**}, \mu)$, so that equity issuance takes place whenever the firm runs out of cash if and only if the profitability prospects are greater than the threshold y_i^*. At issuance dates, the optimal cash reserve process is reflected in a horizontal direction on the (x, y)-plane, the marginal value of cash $V_x^*(0, y)$ is equal to the issuance cost p, and $V^*(0, y) > 0$ for all $y > y_i^*$.*
- *The dividend boundary function b^* is increasing for $y \leq y^{i*}$ and decreasing for $y \geq y^{i*}$, where y^{i*} satisfies $(\phi - b^*)(y^{i*}) = \phi(y_i^*)$. The maximum cash target level, $b^*(y^{i*})$, satisfies $b^*(y^{i*}) > x_\mu$. The dividend boundary function is not differentiable at y^{i*}.*

[19] Recall that y_i^* characterizes the profitability prospects above which the firm issues new shares when the cash reserves are depleted. By construction, we have that $k(z_i^*) = y^{i*} = k(\phi(y_i^*))$ and thus $(\phi - b^*)(y^{i*}) = \phi(y_i^*)$.

Theorem 1 delivers several results. In our model, uncertainty about the firm's profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its long-term profitability. Two opposite effects are at work. First, a positive shock to earnings increases profitability prospects and may induce the firm's management to lower cash target levels because of the cost of accumulating cash. Second, a firm has more to lose from liquidity constraints when profitability prospects are high than when they are low. This may induce the firm's management to accumulate more cash when profitability prospects increase. Theorem 1 shows that the second effect dominates when the firm has no access to capital markets, while the first effect dominates when the firm does have access to capital markets. As a consequence, the dividend boundary b^* is increasing in the profitability prospects when $p \geq \overline{p}$ and is nonmonotonic in the profitability prospects when $p < \overline{p}$. In the latter case, shareholders increase the cash target levels after a new performance record as long as the firm cannot tap the market, that is, as long as the firm's performance is lower than z_i^* (equivalently, as long as the profitability prospects are lower than y_i^* when the cash reserves are depleted). When the cash reserves reach the level $b^*(k(z_i^*)) = b(y^{i*})$, shareholders decide to lower the cash target levels because the firm now has access to the capital markets and has an important amount of cash reserves that is costly to bear. Note that the dividend boundary function converges to the complete information dividend threshold x_μ when the profitability prospects tend to μ. This implies the inequality $b^*(y^{i*}) > x_\mu$. Thus, the firm reaches its higher level of cash reserves at the threshold of market access. At that moment, the firm has a higher cash target level than what would have been optimal in a complete information setting. Another salient feature that we comment on in the next section is the nondifferentiability of b^* at y^{i*}.

When $p < \overline{p}$, in contrast to the complete information benchmark, firms either optimally issue new equity or are liquidated. That is, if the profitability prospects when cash reserves are depleted are above y_i^*, then the firm issues new shares, whereas the firm is liquidated if profitability prospects are lower than y_i^*.[20] Thus, in our model, profitability issues create liquidity issues, and ultimately, the firm is liquidated for liquidity reasons.[21]

[20] This is consistent with the empirical findings of DeAngelo, DeAngelo, and Stulz (2010) that an SEO reflects the corporate life cycle and that, without the offer proceeds, most firms would run out of cash the year after the SEO.

[21] Hugonnier, Malamud and Morellec (2014) provide a model in which a firm sometimes issues new equity and sometimes is liquidated. This feature relies on the occurrence of unpredictable arrivals of financiers. On the contrary, in our model, issuing or not issuing equity is an event that can be anticipated by all participants in the market. This latter feature is also true in the BWY (2019) real options model.

The profitability prospects y^{**} required to run the project are strictly larger than the first-best liquidation threshold y^*. In particular, a negative exogenous shock that leads to profitability prospects below y^{**} triggers liquidation even if cash reserves are abundant.

5 Model Analysis

Despite its two-dimensional nature, our model leads to analytical formulae that allow for a rigorous analysis of the interplay between dynamics of cash holdings and dynamics of profitability prospects. Our analytical formulae also allow simple numerical illustrations that yield additional insights. Figure 1 plots on the (x, y)-plane the dividend boundary function b^* and the curves $z = \phi(y) - x$ that link cash reserves and profitability prospects for a firm's different performance levels, z. It illustrates the joint dynamics of cash and profitability prospects.[22]

The joint dynamics of cash and profitability prospects: an illustration based on Figure 1. Assume that, at date $t = 0$, the cash reserves X_0 and the profitability prospects Y_0 satisfy the equation $Z_0 = \phi(Y_0) - X_0$ with $Z_0 = -0.40$ such that the pair (X_0, Y_0) is on the green curve in Figure 1. The amount $Z_0 = -0.40$ corresponds to the initial value of the performance process $Z_t = \phi(Y_t) - X_t = Z_0 + D_t - \frac{I_t}{p}$. As long as there are neither payments nor issuances, $D_t = I_t = 0$, the performance process Z remains constant. Therefore, the two-dimensional process $(X_t,\ Y_t)$ satisfies $\phi(Y_t) - X_t = Z_0$ and thus evolves on the green curve. If the cash reserves increase to the point of exceeding the dividend boundary function b^*, cash is paid out, the cash reserve process is reflected back in the horizontal direction on the (x, y)-plane, and the performance process increases. If performance records accumulate, the process Z_t will eventually increase to the value $z = 0$, so that the process (X_t, Y_t) will satisfy $\phi(Y_t) - X_t = 0$ and thus will evolve on the blue curve.

Consider now that the cash reserves decrease after a series of negative shocks on cash flows to hit zero. Shareholders then decide whether to issue new shares. They do so whenever the profitability prospects are larger than y_i^* or, equivalently, whenever the performance of the firm is above z_i^*. When the firm issues new shares, the cash reserve process is reflected back in the horizontal direction on the (x, y)-plane, and the performance process decreases accordingly. If, as time passes, cumulative issuances become too large, the process (X_t, Y_t)

[22]To ease comparisons, we use in our simulations the baseline parameters considered in studies in which shareholders face only a liquidity concern; see, for instance, Bolton, Chen and Wang (2011) or DMRV (2011). The parameters r, μ and σ are annualized; σ and μ are expressed in millions of dollars.

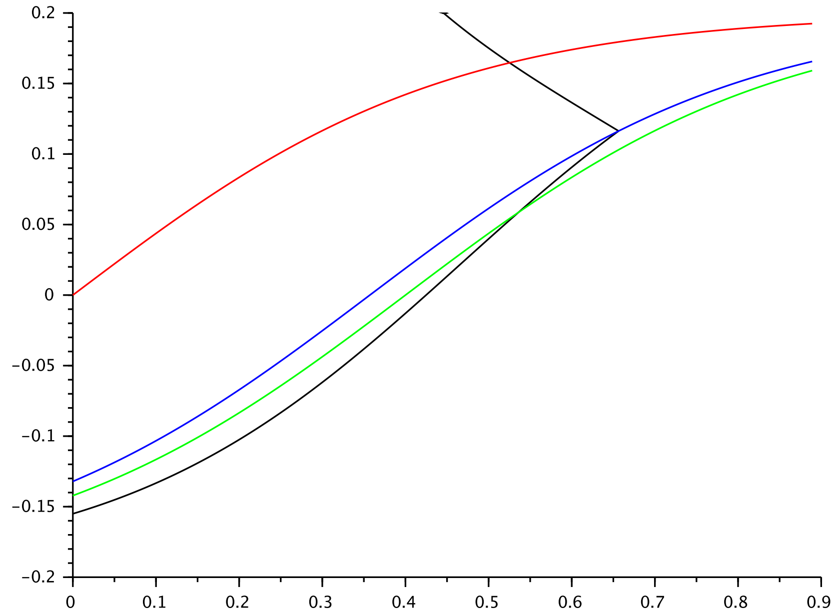

Figure 1: The dividend boundary function b^* (black curve) and three performance level curves $z = \phi(y) - x$ with, from bottom to top, $z = -0.40$, $z = z_i^*$, $z = 0$. The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.5$. For those parameters, $y_i^* = -0.1321$, $z_i^* = -0.3571$, $y^{i*} = 0.1163$, $b(y^{i*}) = 0.6563$ and $x_\mu = 0.44$. The threshold y_i^* corresponds to the level of profitability prospects above which the firm can issue new shares when cash reserves are depleted. Accordingly, $z_i^* = \phi(y_i^*)$ corresponds to the minimum level of performance that allows the firm to issue new shares when cash reserves are depleted, and $y^{i*} = (\phi - b^*)^{-1}(z_i^*)$ corresponds to the profitability prospects at the cash target level when the performance is z_i^*.

will eventually evolve on a constant-performance curve below the blue curve $z_i^* = \phi(y) - x$. The firm then runs the risk of being liquidated because, for such a level of performance, the profitability prospects are too low compared to the financing cost p to allow shareholders to issue new shares when cash reserves are depleted.

Our model results in a two-stage life cycle for the firm: a “probation” stage in which the firm has no access to capital markets and a “mature” stage in which the firm does have access to capital markets. Because of liquidity shocks, the firm can switch from the mature stage to the probation stage, and vice versa. In the probation stage, cash target levels are increasing in the profitability prospects. In the mature stage, the precautionary motive for holding cash weakens, and cash target levels decline when the profitability prospects increase. Thus, our model yields life-cycle stages for the corporate cash dynamics and predicts that the firm's cash holdings are the highest when the firm is at the market access threshold. These findings are consistent with Drobetz, Halling and Schroder (2015), who empirically show the relevance of the corporate life cycle for the dynamics and valuation of cash holdings. They notably find evidence of increases in cash holdings in stages of the life cycle where access to the market is more constrained and of decreases in cash holdings when the firms move toward maturity.

It is worth noting the importance of the initial value of profitability prospects. Two identical firms with the same cash outflow from financing activities, $D_t - \frac{I_t}{p} > 0$, but with different initial profitability prospects and thus different values for Z_0, for example, $Z_0^1 < z_i^* < Z_0^2$, can be in drastically different situations when cash reserves are depleted: it can happen that firm 1 is liquidated because $Z_t^1 = Z_0^1 + D_t - \frac{I_t}{p} < z_i^*$, whereas firm 2 issues new shares because $Z_t^2 = Z_0^2 + D_t - \frac{I_t}{p} \geq z_i^*$. Thus, our model predicts that the initial profitability prospects, which may result, for instance, from financial analysts, have lasting effects on the corporate cash policy. This feature is consistent with the important role of specialized intermediaires in the financing of innovation as, for instance, pointed in Kerr and Nanda (2015).

Figure 1 also illustrates that the dividend boundary function b^* attains its maximum at a kink. The fact that b^* is nondifferentiable at its maximum yields new insights on the corporate propensity to save at cash target levels, as we explain in the next paragraph.

Corporate propensity to save. What does a firm do with a marginal \$1 when it is exactly at its target level of cash? In standard models with only liquidity issues, the firm pays out \$1 as a dividend whatever the importance of financing frictions. This generates the prediction that firms have no propensity to save cash at the cash target level. In contrast, this is not the case in our two-dimensional model in which the propensity to save cash is a function of profitability prospects. This is a unique feature of our model.

Specifically, suppose that the firm is at its cash target level $x = b^*(y)$ and consider what happens after a positive shock to the cash flow. To account for the sign of the change in cash flow over a small period of time h, we consider a $\sqrt{h}$ Euler approximation of the model,

$$X_h = x + \sigma\sqrt{h}B_1,$$

and

$$Y_h = y + \frac{\mu^2 - y^2}{\sigma}\sqrt{h}B_1,$$

where B_1 is a standard Gaussian variable. Therefore, $X_h - x$, the amount of cash available for distribution at time h, and $X_h - b^*(Y_h)$, the amount paid out to shareholders at time h, satisfy

$$\begin{aligned} X_h - b^*(Y_h) &= X_h - (b^*(y) + b^{*'}(y)(Y_h - y)) \\ &= X_h - x - b^{*'}(y)\frac{\mu^2 - y^2}{\sigma}\sqrt{h}B_1 \\ &= (1 - s(y))(X_h - x), \end{aligned}$$

with

$$s(y) = b^{*'}(y)\frac{\mu^2 - y^2}{\sigma}. \tag{27}$$

The function $s(y)$ indicates, for profitability prospects y, the percentage of each dollar earned above the dividend boundary that is reinvested into the firm. The function $s(y)$ corresponds in our framework to the *firm's propensity to save cash out of cash flows* that Almeida, Campello and Weisbach (2004) introduced in their influential paper. The propensity to save measures how much of its current cash flow a constrained firm will save. There is no consensus in the literature about its sign. Almeida, Campello and Weisbach (2004) find evidence that financially constrained firms, because of their limited access to capital markets, have a positive propensity to save. On the other hand, Riddick and Whited (2009) find theoretically and empirically that firms have a negative propensity to save out of cash flow.[23] Décamps, Gryglewicz, Morellec, and Villeneuve (2017) find that the propensity to save can be positive or negative depending on the correlation between temporary shocks and permanent shocks to the cash flows.

Our two-dimensional model delivers novel insights. Our model relates the sign of the propensity to save to the corporate life cycle of the firm. Specifically, formula (27) links the propensity to save to the derivative of the dividend boundary function b^* and yields the following result. In the probation stage, after a new performance record, the firm increases its cash target level and saves cash above cash target levels (the propensity to save is positive). The firm reaches its maximum cash target level at the threshold of access to capital markets for $y = y^{i*}$. In the mature stage, the firm has access to capital markets. It decreases its cash target level after a new performance record and dissaves (the propensity to save is

[23]Riddick and Whited (2009) extend the two-period model of Almeida, Campello, and Weisbach (2004) in several directions. Notably, their model is dynamic and allows capital investment.

negative). The decision to dissave yields a discontinuity in the propensity to save at y^{i*} that originates from the nondifferentiability of b^* at y^{i*}. This break reflects the change in the cash management policy when changing between corporate life-cycle stages. When its performance level allows the firm to enter the mature stage, the cost of holding cash becomes prominent, leading to a rupture in the propensity to save. Let us observe that because of the nonmonotonicity of the dividend boundary function, the cash target level is not a sufficient statistic to infer whether the propensity to save is positive or negative. In line with corporate life-cycle theory, we must know what stage the firm is in to deduce the saving policy from cash target levels. Finally, the propensity to save tends toward 0 when y tends toward μ, reflecting the fact that the model converges to the complete information benchmark.

Numerical simulations provide additional insights. Figure 2 is representative of our numerical simulations and plots the propensity to save for proportional issuance costs $p = 1.5$ and $p = 1.05$ (with other baseline parameters remaining the same). In the probation stage, the propensity to save is decreasing. In the mature stage, it is increasing for large issuance costs (left). For low issuance costs (right), the propensity to save is slightly decreasing when the firm enters in its mature phase, and then it is increasing. Overall, our model predicts a negative relationship between cash target levels and the corporate propensity to save cash.

Figure 2 also shows that the propensity to save is increasing in the cost of external financing in the probation stage. We also get the prediction that a firm in a costly issuance environment dissaves more aggressively in its mature stage than a firm in a cheap issuance environment. The reason is that both the firm's optimal level of cash and required profitability prospects to access the mature stage increase with the cost of external financing. It follows that a high-cost firm has more slack with which to respond to positive shocks to earnings and dissaves more aggressively in the mature stage to counteract the cost of holding cash. Specifically, for $p = 1.5$, the maximum cash target level is $b(y^{i*}) = 0,6563$, far above the cash target level $x_\mu = 0,4457$ that prevails in the complete information case. The propensity to save is around -80% for profitability prospects just above the threshold $y^{i*} = 0,1163$. For $p = 1.05$ the maximum cash target level $0,2358$ is slightly higher than the constant cash target level of the complete information case $x_\mu = 0,1829$. The propensity to save is around -5% for profitability prospects just above the threshold $y^{i*} = -0,1025$.

So far, we have focused on the propensity to save. It is worth noting that $1 - s(y)$ corresponds to the percentage of each dollar earned above the dividend boundary that is distributed to shareholders in the form of cash. Therefore, the amount $1 - s(y)$ can be interpreted as the payout ratio function of the firm. In standard models with no uncertainty

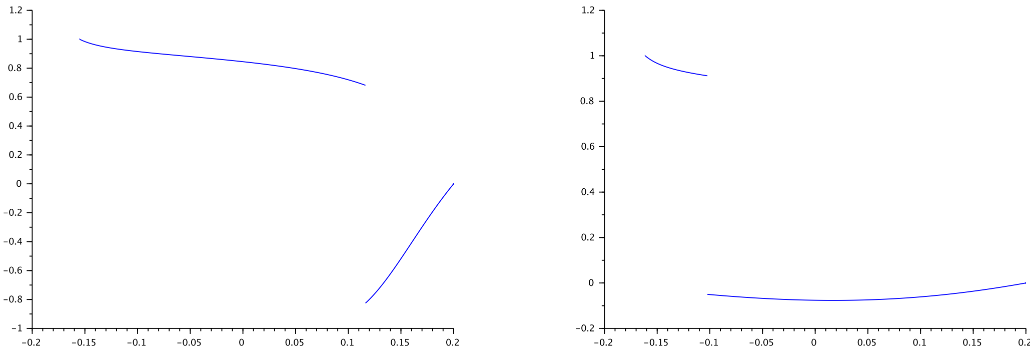

Figure 2: The propensity to save as a function of profitability prospects. (Left) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.5$. (Right) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.05$

about long-term profitability, the payout ratio is 100%, whatever the costs of external financing. Thus, in our two-dimensional model, a dividend decision is made when cash reserves reach the cash target level $b^*(y)$. Shareholders then receive $1 - s(y)$ percent of the cash above $b^*(y)$.

Figure 3 depicts the mapping $y \longrightarrow 1 - s(y)$ and delivers additional insights. Our model suggests that a cash-constrained firm with uncertain long-term profitability pays very little in dividends as long as it has no access to the market. This effect is even more true when the cost of external financing is high.

At the market access threshold, the firm has a significant pile of cash and high profitability prospects (both of these levels are all the higher because external financing costs are high). This provides the firm with the ability to generate and sustain free cash flows. Once profitability prospects are sufficiently high, it is then optimal to initiate dividend payments. This causes a jump in the payout ratio function (equivalently, a jump in the propensity to save, as already explained). This result is in line with DeAngelo, DeAngelo and Stulz (2006), whose analysis suggests that, if well-established firms had not paid dividends as observed, their cash balances would be enormous, thus granting extreme discretion to managers of these firms.

Finally, let us observe that the jump in the payout ratio is not related to a jump in firm value, which continuously evolves as a function of cash reserves and profitability prospects. Our model thereby provides an example in which the jump of a key indicator of the firm's governance is not signaling. There are no asymmetries of information in our model. All our findings result from the (endogenous) dynamics of the interaction between learning about long-term profitability and the cost and benefit of holding cash.

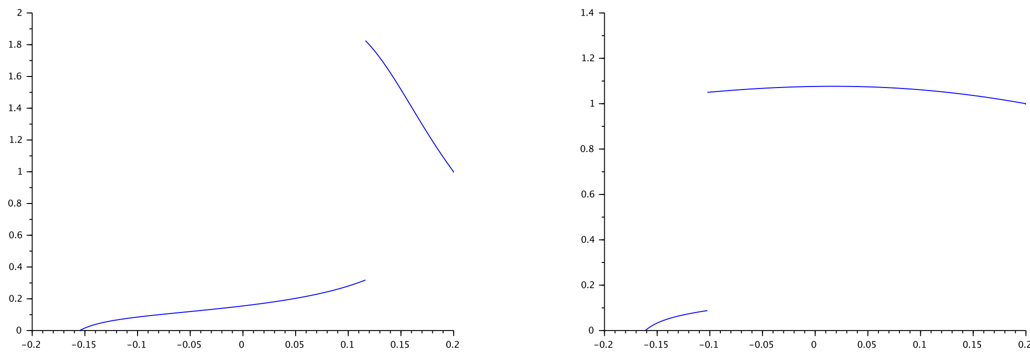

Figure 3: The payout ratio as a function of profitability prospects. (Left) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.5$. (Right) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.05$

Firm dynamics. Our model also delivers new insights into the dynamics of firm value across life stages. We first consider the firm value process between two consecutive dates of issuance decision and payment decision. We saw that for a given performance $z \in [\phi(y^{**}), \infty)$, payment occurs at time $\tau_z \equiv \{t \geq 0 \,|\, X_t = b^*(k^*(z))\}$, and issuance (or liquidation) at time $\tau_0 = \{t \geq 0 \,|\, X_{t^-} = 0\}$. Thus, the firm value process can be written on time interval $[0, \tau_0 \wedge \tau_z]$ as a function of the cash reserve process $\{X_t,\, t \geq 0\}$. We use the change of variable (17) to obtain that $V^*(x, y) = V^*(x, \psi(x + z))$, and we denote $W^*(x, z) \equiv V^*(x, \psi(x + z))$. Then, by applying Itô's formula to the process $\{W^*(X_t, z),\ t \geq 0\}$, we easily obtain the following proposition.

Proposition 3 *For a given performance, $z \geq \phi(y^{**})$, the mapping $x \longrightarrow W^*(x, z)$ is increasing on $[0, b^*(k^*(z))]$. The firm value process $\{W^*(X_t, z),\ t \geq 0\}$ satisfies, for any $t \in [0, \tau_0 \wedge \tau_z]$, the dynamics*

$$dW^*(X_t, z) = rW^*(X_t, z)\, dt \ + \ \sigma\, W^*_x(X_t, z)\, dB_t,$$

with

$$\sigma W^*_x(x, z) = \sigma V^*_x(x, \psi(x + z)) + \sigma \psi'(x + z) V^*_y(x, \psi(x + z)), \tag{28}$$

so that the volatility of the firm at cash target $b^*(k^*(z))$ satisfies

$$\sigma\, W^*_x(b^*(k^*(z)), z) = \sigma + \frac{\mu^2 - k^*(z)^2}{\sigma} V^*_y(b^*(k^*(z)), k^*(z)). \tag{29}$$

The mapping $x \longrightarrow W^*(x,z)$ represents the value of the firm as a function of the cash reserves for a fixed level of performance z.[24] It corresponds to the value of the firm between two consecutive dates of issuance decision and payment decision. Formulae (28) and (29) give the volatility of the firm as a function of the cash reserves.

For a fixed performance z, an increase in cash also increases profitability prospects so that the relation $x - \phi(y) = z$ holds true between two consecutive dates of issuance decision and payment decision. Then, two effects are at work. First, an increase in cash diminishes the precautionary motive for holding cash. The marginal value of cash decreases in cash levels. Accordingly, one more unit of cash has a larger impact on the value of the firm when the latter is initially low than when it is high. Formally, the mapping $x \longrightarrow V^*(x,y)$ is concave, as proven in the Appendix.[25] Second, an increase in cash increases profitability prospects. The ongoing project is more likely to have a positive long-term profitability. Accordingly, one more unit of profitability prospects has a larger impact on the value of the firm when the latter is initially high than when it is low. Formally, the mapping $y \longrightarrow V^*(x,y)$ is convex.

Our numerical study shows that the first effect dominates when the firm has proven access to the market, in the sense that its performance z is sufficiently above the market access threshold z_i^*. The second effect, which we call the learning effect, dominates when the firm is fully in the probation stage in the sense that its performance z is sufficiently below z_i^*. Close to the market access threshold (when z is around z_i^*), both effects are present. Specifically, Figure 4 illustrates that the mapping $x \longrightarrow W^*(x,z)$ is concave for z values sufficiently above z_i^*, convex for z values sufficiently below z_i^*, and convex then concave for z values around z_i^*.

Additionally, Figure 4 illustrates that the mapping $z \longrightarrow W^*(x,z)$ is increasing, as we show in the Appendix. A direct implication follows: Because the firm's performance process is increasing in cumulative payments and decreasing in the cumulative issuances, our two-dimensional model predicts that, by holding liquid assets constant, there is a positive relationship between payments and firm value and a negative relationship between issuances and firm value.

Now, we turn to the volatility of the firm. Figure 5 depicts, for different issuance costs, the volatility of the firm as a function of the cash reserves with different levels of performance z (Equation (28)). In line with the previous observations, the volatility of the firm is decreasing in the cash reserves for z sufficiently above z_i^* and increasing in the cash reserves for z sufficiently below z_i^*. Close to the market access threshold (when z is around z_i^*), the

[24]The amount $\phi(y^{**})$ is the minimum level of performance required by shareholders to run the firm.

[25]See Proposition 11.

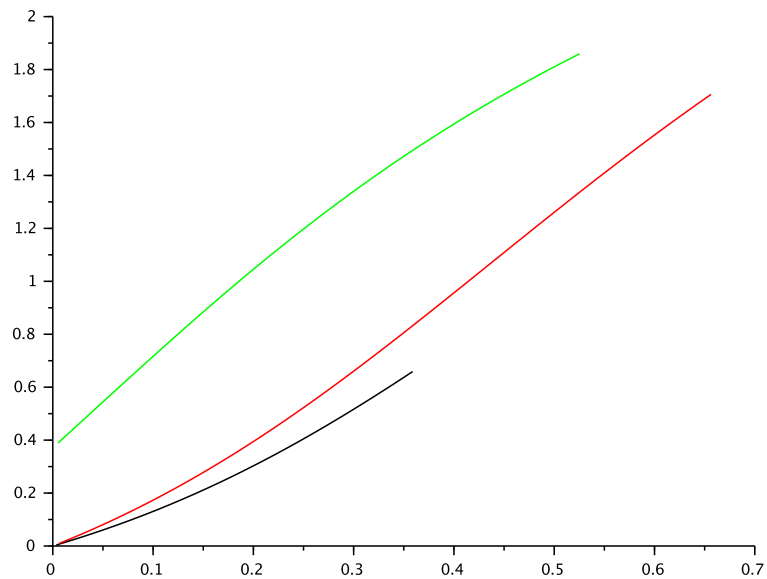

Figure 4: The value of the firm as a function of cash reserves for three levels of performance, $z = -0.4$ (black curve), $z = z_i^*$ (red curve) and $z = 0$ (green curve). The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$ and $p = 1.5$.

volatility of the firm is increasing and then decreasing in the cash reserves.

Noting that the mapping $x \longrightarrow W^*(x, z)$ is increasing, we obtain new testable results on the relationship between the value of the firm and its volatility across firm life-cycle stages. Our model predicts that, for firms with proven access to the market, we should observe a negative relationship between firm value and volatility. For firms still far from accessing the market, we should observe a positive relationship between value and volatility. The negative relationship between the firm's value and its volatility is a standard feature of corporate cash models with complete information.[26] The positive relationship between the volatility of the firm and its value is a well-known feature of corporate models with deep-pocketed shareholders[27] and holds true in our first-best benchmark. Our study shows that the result still holds for cash-constrained shareholders who learn about long-term profitability and have no access to the financial markets. Close to the market access threshold, the volatility of the firm is an inverted U-shaped function of the value of the firm, reflecting the two effects.

[26]See Bolton, Chen, and Wang (2011) and DMRV (2011), among many others.

[27]We refer to the literature initiated by Merton (1973) and Leland (1994).

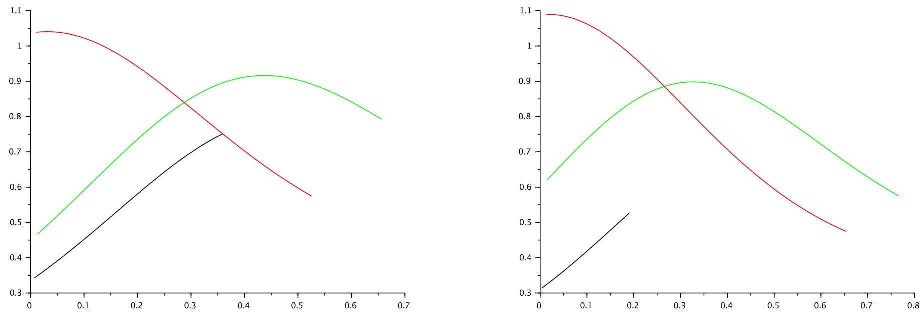

Figure 5: The volatility of the firm as a function of cash reserves for two different levels of issuance costs. (Left) The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$, $z = -0.4$ (black curve), $z = z_i^*$ (green curve) and $z = 0$ (red curve) and $p = 1.5$. (Right) The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$, $z = -0.4$ (black curve), $z = z_i^*$ (green curve) and $z = 0$ (red curve) and $p = 2$.

Therefore, our model suggests that the relationships between the value of the firm and its volatility can drastically change in transition phases between life-cycle stages.

In models with complete information, the volatility of the firm corresponds to the volatility of the cash flows times the marginal value of cash (that is, $\sigma V'_\mu(x)$ with the notations of Proposition 2). The volatility of the firm is more involved in our two-dimensional model. The novel insight is the learning effect that links the volatility of the firm and the marginal value of profitability prospects (that is, the second term of the right-hand side of (28)). Equation (29) highlights that the volatility of the firm at cash target levels is larger than the volatility of cash flows, whereas these volatilities coincide in the complete information benchmark.[28]

Two additional insights on the firm's volatility follow from our model. First, we observe in Figure 5 that the volatility of the firm when cash reserves are depleted is increasing in the performance level z. Thus, the model predicts that the higher the profitability prospects, the higher the volatility of the firm is at issuance dates. Second, at cash target levels, the firm reaches its higher level of volatility when it is at the threshold of access to capital markets, that is, for $z = z_i^*$. This is consistent with the prediction that the dynamics of cash holdings change drastically when the firm's performance crosses the threshold z_i^*.

[28]Technically, at the limit, for $z = \infty$, the model coincides with the complete information benchmark. We show in the Appendix (see Propositions 7 and 11) that $\lim_{z \longrightarrow \infty} W^*(x, z) = V_\mu(x)$ and that $\lim_{z \longrightarrow \infty} W_x^*(x, z) = V'_\mu(x)$. In particular, we have the equality $\sigma \lim_{z \longrightarrow \infty} W_x^*(b^*(k^*(z)), z) = \sigma$.

6 Conclusion

We study the corporate cash management of an all-equity firm that is conducting a project whose long-term profitability is unknown and that faces financing constraints. We show that the trade-off between the benefits and the costs of holding cash evolves as shareholders learn about profitability and yields a two-stage life-cycle for the firm: a probation stage and a mature stage. The firm can go back and forth between the two stages. In the mature stage, the firm issues new shares when cash is exhausted. In the probation stage, the firm has no access to capital markets and is liquidated when cash reserves are depleted. Issuing new shares requires a sufficiently high level of profitability prospects. A payment decision is made when the total flow of cash into the firm reaches a target level, which depends on the profitability prospects. Shareholders reinvest into the firm a proportion of cash above the cash target level and pay out the complement as dividends. Saving and payout decisions are functions of the profitability prospects. Cash target levels increase in the probation stage and decrease in the mature stage. Payout ratios are low in the probation stage and high in the mature stage. Thus, the model offers a rationale for stylized facts stemming from the life-cycle theory of the firm. Young firms hold more cash and pay little in dividends, while firms with proven profitability are more prone to pay dividends and decrease their cash target levels.

Our study goes beyond these implications and delivers new predictions that call for additional theoretical and empirical studies. We obtain that, the propensity to save cash at cash target levels is positive and decreasing in the profitability prospects in the probation stage, while it is the opposite (negative and increasing) in the mature stage. The model predicts that firms reach their maximum level of cash reserves at the market access threshold. At that moment, the firm's volatility also reaches a maximum value. In the probation stage, the model predicts a positive relationship between the firm's value and its volatility, while the relationship is negative for firms with proven access to the market. Close to the market access threshold, the volatility of the firm is an inverted U-shaped function of the value of the firm. When the costs of external financing increase, both the profitability prospects necessary to access to the market and the cash target levels increase. This implies that when firms access the market, they dissave more aggressively when costs of external financing are high than in an environment with low costs of external financing. Overall, our model suggests that corporate cash policy and the dynamics of firm value drastically change in transition phases between life-cycle stages. Additionally, the model suggests that the initial assessment of the firm's long term profitability has long lasting consequences on its corporate cash management and matters for evaluating its performance at any future date.

As explained in the introduction, our model adresses the case of all-equity young firms that have little collateral to offer. Well established firms with more elaborated financial structure face also corporate cash management issues, and learn about their long term profitability when they launch new projects or engage in major restructurings. Clearly, future learning models should integrate into the analysis a wider range of financial tools, especially debt issuance and the use of credit lines. We lack of studies on the joint evolution of cash holdings, debt, and profitability prospects. In addition, note that other learning issues arise in corporate finance. For instance, some studies focus on learning about the state of the economy, or learning about a rival's charateristics, nevertheless avoiding cash management considerations.[29] Here again we lack of studies that integrate these learning issues in a setting of constrained financing. These and related questions must await for future work.

[29]See for instance, Grenadier and Malenko (2010), Décamps and Marioti (2004).

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8 Appendix

8.1 First-best benchmark

Proposition 1 relies on the following Lemma.

Lemma 1 *The value of the firm V^* can be written in the form*

$$V^*(x,y) \;=\; x + \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\left(Y_s - rX_s\right)\,ds\right] \le x + \frac{\mu}{r}. \tag{30}$$

Proof of Lemma 1. For all $t>0$ and all admissible controls I and D, we have

$$e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} = x + \int_0^{t\wedge\tau_0} e^{-rs}dX_s - r\int_0^{t\wedge\tau_0} e^{-rs}X_s\,ds.$$

Because X_t is nonnegative for $t\le\tau_0$ and Y is bounded above by μ, we obtain

$$\begin{aligned} e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} + \int_0^{t\wedge\tau_0} e^{-rs}(dD_s - dI_s) \;&=\; x + \int_0^{t\wedge\tau_0} e^{-rs}dR_s - r\int_0^{t\wedge\tau_0} e^{-rs}X_s\,ds \qquad (31)\\ &\le\; x + \frac{\mu}{r} + \sigma\int_0^{t\wedge\tau_0} e^{-rs}dB_s. \end{aligned}$$

Applying the optional sampling theorem to the uniformly $\mathcal{F}^B$-martingale $\left(\int_0^t e^{-rs}dB_s\right)_{t\ge 0}$, we obtain after taking expectations,

$$\mathbb{E}\left[e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} + \int_0^{t\wedge\tau_0} e^{-rs}(dD_s - dI_s)\right] \le x + \frac{\mu}{r}.$$

Letting t goes to $+\infty$, we obtain for all $(x,y)\in[0,\infty)\times(-\mu,\mu)$

$$V(x,y;I,D) \le x + \frac{\mu}{r} < \infty.$$

Thus, the value function V^* is finite. We obtain (30) from equations (3) and (31). □

Proof of Proposition 1. Let us define

$$\Gamma(y) = \sup_{\tau\in\mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs}Y_s ds\right].$$

Standard results in optimal stopping theory[30] yield that the optimal value function Γ is C^1 on $[0,\infty)$ and that a threshold strategy $\tau^* = \inf\{t\ge 0\,|\,Y_t = y^*\}$ is optimal. The value function Γ can be written in terms of the free boundary problem:

$$\begin{cases} \frac{1}{2\sigma^2}(y+\mu)^2(\mu-y)^2\Gamma''(y) - r\Gamma(y) + y = 0,\; y\ge y^*,\\ \Gamma(y^*) = 0,\;\; \Gamma'(y^*) = 0. \end{cases}$$

[30] See for instance Peskir and Shiryaev (2006).

Standard computations yield

$$\begin{cases} \hat{V}(x,y) = x, \quad -\mu < y \leq y^*, \\ \hat{V}(x,y) = x + \frac{y}{r} - \frac{h_1(y)}{h_1(y^*)}\frac{y^*}{r} \quad y^* \leq y < \mu, \end{cases} \tag{32}$$

where

$$h_1(y) = (\mu + y)^{\gamma}(\mu - y)^{1-\gamma}, \; y^* = \frac{-\mu}{1-2\gamma} < 0, \tag{33}$$

$$\text{and where } \gamma \text{ is the negative root of the equation } x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0. \tag{34}$$

It follows from (32), (33) that $\hat{V}(x,y)$ is increasing and convex in y.

Proof of Corollary 1. Because the cash reserves are positive for all admissible controls, it follows from (30) that,

$$x + \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} Y_s \, ds\right]$$

is an upper bound for V^*. From Equation (3), any admissible control (I,D) acts on $\mathbb{E}\left[\int_0^{\tau_0} e^{-rs} Y_s \, ds\right]$ by modifying only the $\mathcal{F}^R$-stopping time τ_0. Thus,

$$\sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} Y_s \, ds\right] \leq \sup_{\tau\in\mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs} Y_s ds\right],$$

which implies that the function V^* is bounded above by $\hat{V}$. □

8.2 Complete information benchmark.

In this section, we consider that the firm's long-term profitability is known and is equal to μ. We develop the mathematical formulation of Proposition 2. This formulation yields useful formulae for the proof of Theorem 1. We prove also in this section the Corollary 2.

8.2.1 No equity issuance

We start the analysis when security issuances are not allowed. Then, the dynamics of the cash reserve satisfy

$$dX_t = \mu \, dt + \sigma dB_t - dD_t,$$

and thus the shareholders' problem writes

$$\overline{V}_\mu(x) = \sup_{D\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} dD_s\right], \tag{35}$$

where $\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}$. The following result is due to Jeanblanc and Shiryaev (1995).

Proposition 4 *The value function $\overline{V}_\mu$ of problem (35) is concave, twice continuously differentiable and it satisfies the following HJB equation on $(0, +\infty)$:*

$$\max\left\{\frac{\sigma^2}{2}\overline{V}''_\mu + \mu\overline{V}'_\mu - r\overline{V}_\mu, 1 - \overline{V}'_\mu\right\} = 0.$$

Moreover, we have

$$\begin{cases} \overline{V}_\mu(x) = \dfrac{e^{-\beta\gamma x} - e^{\beta(\gamma-1)x}}{-\beta\gamma e^{-\beta\gamma\overline{x}_\mu} + (1-\gamma)\beta e^{\beta(\gamma-1)\overline{x}_\mu}} & 0 \leq x \leq \overline{x}_\mu, \\ \overline{V}_\mu(x) = x - \overline{x}_\mu + \frac{\mu}{r}, \quad x \geq \overline{x}_\mu, \end{cases} \tag{36}$$

with

$$\overline{x}_\mu = \frac{1}{\beta(1-2\gamma)} \ln(\frac{1-\gamma}{\gamma})^2 = 2\frac{y^*}{\mu}\phi(y^*), \tag{37}$$

where (33) and (34) define y^* and γ and where $\beta = \frac{2\mu}{\sigma^2}$. Any excess of cash over the dividend boundary $\overline{x}_\mu$ is paid out to shareholders, such that the cash reserve process is reflected back each time it reaches $\overline{x}_\mu$. The process $D = \{D_t; t \geq 0\}$ with

$$D_t = (x - \overline{x}_\mu)^+ \mathbb{1}_{t=0} + L_t^{\overline{x}_\mu} \mathbb{1}_{t>0} \tag{38}$$

is the optimal dividend payment process. In equation (38), $L^{\overline{x}_\mu}$ denotes the so-called local time process solution to the Skohorod problem[31] at $\overline{x}_\mu$ for the drifted Brownian motion $\mu t + B_t$.

8.2.2 Equity issuance

When security issuances are allowed at a proportional issuance cost $p > 1$, the dynamics of the cash reserves satisfy

$$dX_t = \mu\, dt + \sigma dB_t - dD_t + \frac{dI_t}{p},$$

and shareholders' problem writes

$$V_\mu(x) = \sup_{I,D\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}(dD_s - dI_s)\right] \tag{39}$$

where $\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}$. The following proposition is due to Lokka and Zervos (2008) and provides a rigorous formulation of Proposition 2 in the main text.

[31] See Karatzas and Shreve (1991) page 210.

Proposition 5 *The value function defined in (39) is concave, twice continously differentiable and it satisfies the following HJB equation on* $(0, +\infty)$*:*

$$\max\{\frac{\sigma^2}{2}V_\mu'' + \mu V_\mu' - rV_\mu, 1 - V_\mu', V_\mu' - p\} = 0.$$

Moreover, we have

- *If* $p \geq \overline{V}_\mu'(0)$ *then,* $V_\mu(x) = \overline{V}_\mu(x)$ *for all* $x \geq 0$*.*
- *If* $p < \overline{V}_\mu'(0)$ *then,*

$$\begin{cases} V_\mu(x) = \dfrac{1-\gamma}{\beta\gamma}\dfrac{y^*}{\mu}e^{-\beta\gamma(x-x_\mu)} + \dfrac{\gamma}{\beta(\gamma-1)}\dfrac{y^*}{\mu}e^{\beta(\gamma-1)(x-x_\mu)} & 0 \leq x \leq x_\mu, \\ V_\mu(x) = x - x_\mu + \frac{\mu}{r}, \quad x \geq x_\mu, \end{cases} \tag{40}$$

where x_μ *is defined as the unique solution to the equation*

$$p = -\frac{y^*}{\mu}(1-\gamma)e^{\gamma\beta x_\mu} + \frac{y^*}{\mu}\gamma e^{(1-\gamma)\beta x_\mu}. \tag{41}$$

Any excess of cash over the dividend boundary x_μ *is paid out to shareholders, so that the cash reserve process is reflected back each time it reaches* x_μ*. There is equity issuance whenever the firm runs out of cash, so that the cash reserve process is reflected back each time it reaches* 0*. The processes* $D = \{D_t; t \geq 0\}$ *and* $I = \{I_t; t \geq 0\}$ *with*

$$D_t = (x - x_\mu)^+ \mathbb{1}_{t=0} + L_t^{x_\mu}\mathbb{1}_{t>0} \;\; \text{and} \;\; I_t = L_t^0 \mathbb{1}_{t>0} \tag{42}$$

are the optimal dividend payment and equity issuance processes. In equation (42), L^{x_μ} *and* L^0 *denote the solution to the Skohorod problem at* x_μ *and at* 0 *for the drifted Brownian motion* $\mu t + B_t$*.*

Hereafter, we will note $\overline{p} = \overline{V}_\mu'(0)$. The thresholds $\overline{x}_\mu$ and x_μ defined in (37) and (41) satisfy $\overline{x}_\mu > x_\mu$. Moreover, equation (36) yields that $\overline{p} = \frac{1-\gamma}{-\gamma}e^{\gamma\beta\overline{x}_\mu}$. We deduce that for $1 < p < \overline{p}$, we have

$$p < \frac{1-\gamma}{-\gamma}e^{\beta\gamma x_\mu}. \tag{43}$$

We will use later this inequality.

Proof of Corollary 2. We use equation (30) and the fact that the process Y_t is bounded by μ to get

$$V^*(x, y) \leq x + \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}(\mu - rX_s)\, ds\right].$$

Proceeding analogously as in the proof of Lemma 1, we have

$$V_\mu(x) = x + \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}(\mu - rX_s)\,ds\right],$$

which leads to $V(x,y) \le V_\mu(x)$. □

8.3 Model Solution

We devote this section to the proof of Theorem 1. We use the following notations and relationships throughout the proof.

- The function $h_1(y) = (y+\mu)^\gamma(\mu-y)^{1-\gamma}$ is decreasing and convex over $(-\mu,\mu)$ and

$$h_1'(y) = h_1(y)(\frac{\mu^2}{y^*} - y)\frac{1}{\mu^2-y^2}, \quad h_1''(y) = \frac{2r\sigma^2}{(\mu^2-y^2)^2}h_1(y). \tag{44}$$

- The function $h_2(y) = (y+\mu)^{1-\gamma}(\mu-y)^\gamma$ is increasing and convex over $(-\mu,\mu)$ and

$$h_2'(y) = h_2(y)(-\frac{\mu^2}{y^*} - y)\frac{1}{\mu^2-y^2}, \quad h_2''(y) = \frac{2r\sigma^2}{(\mu^2-y^2)^2}h_2(y). \tag{45}$$

- For all $y \in (-\mu,\mu)$

$$h_1(y)h_2(y) = \mu^2 - y^2, \text{ and } h_1'(y)h_2(y) - h_2'(y)h_1(y) = \frac{2\mu^2}{y^*}. \tag{46}$$

- The function $\psi(z) = \phi^{-1}(z) = \mu\frac{e^{\beta z}-1}{e^{\beta z}+1}$ with $\beta = \frac{2\mu}{\sigma^2}$, is increasing over $\mathbb{R}$ and

$$\psi'(z) = \frac{\mu^2 - \psi(z)^2}{\sigma^2}. \tag{47}$$

- For all $z \in \mathbb{R}$,

$$h_1(\psi(z)) = 2\mu e^{(\gamma-1)\beta z}\frac{1}{e^{-\beta z}+1}, \quad h_2(\psi(z)) = 2\mu e^{-\gamma\beta z}\frac{1}{e^{-\beta z}+1}. \tag{48}$$

As for the complete information benchmark, it is useful to start the analysis under the assumption that the shareholders are not allowed to issue new shares.

8.3.1 No equity issuance

Thus, in this subsection we solve the problem

$$\overline{V}(x,y) = \sup_{D\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right], \tag{49}$$

where $\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}$ with $X_t = \phi(Y_t) - \phi(y) + x - D_t$.

As a preliminary but essential step, we establish a standard verification Lemma that specifies conditions under which a function V defined on $[0,\infty) \times (-\mu,\mu)$ is a majorant of the value function $\overline{V}$ of the problem (49).

Lemma 2 *(Verification Lemma) Assume there exists a function V defined on $[0,\infty) \times (-\mu,\mu)$ that satisfies*

1. *V is twice differentiable,*
2. *V has bounded first derivatives,*
3. *$V(0,y) = 0$ for all $y \in (-\mu,\mu)$ and*

$$\max(\mathcal{L}V - rV, 1 - V_x) \leq 0 \text{ on } [0,\infty) \times (-\mu,\mu),$$

then V is a majorant of $\overline{V}$.

Proof of Lemma 2. See the online appendix. □

Second, we explicitly build such a majorant. To this end, we prove that the following variational problem

$$\begin{aligned}
\mathcal{L}V(x,y) - rV(x,y) &= 0 \quad \text{on the domain } \{(x,y),\ 0 < x < b(y),\ -\mu < y < \mu\}, && (50)\\
V(0,y) &= 0 \quad \forall y \in (-\mu,\mu), && (51)\\
V_x(x,y) &= 1, \text{ for } x \geq b(y), && (52)\\
V_{xy}(b(y),y) &= 0, && (53)\\
\lim_{y\longrightarrow\mu} V(x,y) &= \overline{V}_\mu(x) \ \ \forall x \geq 0, && (54)
\end{aligned}$$

has a unique solution (V,b) (see Proposition 6), such that V satisfies Lemma 2 (see Proposition 7) and thus dominates $\overline{V}$. Finally, we show in Proposition 8 that V can be reached by an admissible policy and thus coincides with the solution $\overline{V}$ to the problem (49). This last step also provides the optimal dividend policy and concludes the study of problem (49).

We start with a technical lemma.

Lemma 3 *The ordinary differential equation*

$$g'(y) = f(g(y), y), \tag{55}$$
$$g(\mu) = \overline{x}_\mu, \tag{56}$$

with

$$f(x,y) = \frac{\sigma^2}{\mu^2 - y^2} \frac{yy^* + \left(\mu - r\sigma^2 \left(\frac{y^*}{\mu}\right)^2 \frac{1}{\mu}\right) \phi^{-1}\left(-\frac{\mu}{y^*}\, x\right)}{yy^* \, + \, \mu\phi^{-1}\left(-\frac{\mu}{y^*}\, x\right)} \tag{57}$$

defined on the domain $\{(x,y) \in [0,\infty) \times (-\mu,\mu) \,|\, x > \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu})\}$ *has a unique solution. The solution* g *is* C^1 *and increasing over* $[\overline{y}^{**}, \mu]$ *where the threshold* $\overline{y}^{**} \equiv g^{-1}(0)$ *is well defined and strictly larger than* y^**.*

Moreover, if we define $\overline{b} = \max(g, 0)$*, then* $\overline{k} = (\phi - \overline{b})^{-1} : [\phi(\overline{y}^{**}), \infty) \longrightarrow [\overline{y}^{**}, \mu)$ *is a well defined* C^1 *increasing function. The function* $\overline{k}$ *is the unique solution to the ordinary differential equation*

$$\overline{k}'(z) = \Theta(z, \overline{k}(z)), \tag{58}$$
$$\lim_{z \longrightarrow \infty} \phi(\overline{k}(z)) - z = \overline{x}_\mu, \tag{59}$$

with

$$\Theta(z,y) = \frac{\mu^3}{y^* r\sigma^2} \frac{\mu^2 - y^2}{\sigma^2} \frac{\psi\left(\frac{\mu}{y^*}(\phi(y) - z)\right)\mu - yy^*}{y^*\psi\left(\frac{\mu}{y^*}(\phi(y) - z)\right)} \tag{60}$$

defined on the domain $\{(z,y) \in \mathbb{R} \times (-\mu,\mu) \,|\, \phi(y) - z > \max(0, \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu}))\}$.

Proof of Lemma 3. See the online appendix $\square$

We are now in a position to solve the system (50)-(54).

Proposition 6 *Let us consider the functions* $\overline{b}$ *and* $\overline{k}$ *defined in Lemma 3 and the function* $(x,y) \longrightarrow V(x,y)$ *defined on* $[0,\infty) \times (-\mu,\mu)$ *by the relations*

$$\begin{cases} V(0,y) = 0, \quad \text{for } y \in (-\mu,\mu), \\ V(x,y) = A\left(\phi(y) - x\right)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-x)}h_2(y)\right), \quad \text{for } 0 \le x \le \overline{b}(y), y \in (-\mu,\mu), \\ V(x,y) = x - \overline{b}(y) + V(\overline{b}(y), y), \quad \text{for } x \ge \overline{b}(y), y \in (-\mu,\mu), \end{cases} \tag{61}$$

where

$$A(z) = \frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(\overline{k}(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z} - h_2'(\overline{k}(z))\right). \tag{62}$$

Then, the couple $(V, \overline{b})$ *is the unique solution to the system (50)-(54). Furthermore, the function* $\overline{b} : [\overline{y}^{**}, \mu] \longrightarrow [0, \overline{x}_\mu]$ *is* C^1 *and increasing.*

Proof of Proposition 6. Having in mind the change of variable (17), we are looking for a smooth function U defined on $[0,\infty)\times(-\mu,\mu)$ and a C^1 function $k:\mathbb{R}\longrightarrow(-\mu,\mu)$ that solve the variational system

$$\begin{aligned}
\frac{1}{2\sigma^2}(\mu^2-y^2)^2U_{yy}(z,y)-rU(z,y) &= 0 \text{ on } \{(z,y),\ z\in\mathbb{R},\ \psi(z)<y<k(z)\}, && (63)\\
U(z,\psi(z)) &= 0, \text{ for } z\in\mathbb{R}, && (64)\\
U_z(z,y) &= -1, \text{ for } k(z)\leq y, && (65)\\
U_{xy}(z,k(z)) &= 0, && (66)\\
\lim_{z\longrightarrow\infty} U(z,\psi(x+z)) &= \overline{V}_\mu(x). && (67)
\end{aligned}$$

First, we establish a set of necessary conditions for the existence of such a pair (U,k) by observing that any solution to the o.d.e. (63) can be written in the form

$$U(z,y)=A(z)h_1(y)+B(z)h_2(y), \tag{68}$$

where (44) and (45) define h_1 and h_2. Using (68), we obtain from (65) and (66) that

$$A'(z)=h_2'(k(z))\frac{y^*}{2\mu^2},\ \text{ and }\ B'(z)=-h_1'(k(z))\frac{y^*}{2\mu^2}. \tag{69}$$

Using again (68), we rewrite (64) in the form

$$A(z)=-B(z)\frac{h_2(\psi(z))}{h_1(\psi(z))}=-B(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}. \tag{70}$$

Taking the derivative of (70), we obtain

$$A'(z)=-B'(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}+B(z)\frac{2}{\sigma^2}\frac{\mu^2}{y^*}e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}$$

which yields using again (70),

$$A(z)=-\frac{\sigma^2y^*}{2\mu^2}\left(A'(z)+B'(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}\right). \tag{71}$$

Using (68) and (70), and plugging (69) into (71) yield that,

$$U(z,y)=A(z)\left(h_1(y)-e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(z)}h_2(y)\right), \tag{72}$$

where

$$A(z) = \frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z}-h_2'(k(z))\right). \tag{73}$$

Taking the derivative of (71) and using (69), we obtain that

$$k'(z) = \frac{2\mu^2}{\sigma^2 y^*} \frac{h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z))}{h_1''(k(z))h_2(\psi(z)) - h_2''(k(z))h_1(\psi(z))}. \tag{74}$$

A computation based on formulae (44)-(48) yields that

$$k'(z) = \frac{\mu^3}{y^* r \sigma^2} \frac{\mu^2 - k(z)^2}{\sigma^2} \frac{\psi\left(\frac{\mu}{y^*}(\phi(k(z)) - z)\right)\mu - k(z)y^*}{y^*\psi\left(\frac{\mu}{y^*}(\phi(k(z)) - z)\right)}, \tag{75}$$

which is positive on the domain $\phi(k(z)) - z > \max(0, \frac{y^*}{\mu}\phi(\frac{y^*}{\mu}k(z)))$.

Thus, (72), (73) and (75) is a set of necessary conditions for the existence of a smooth solution (U, k) to (63), (64), (65), (66). It remains to find a necessary condition for a solution to satisfy (67). Below we prove that U satisfies (67) if and only if $\lim_{z \longrightarrow \infty} \phi(k(z)) - z = \overline{x}_\mu$. From Lemma 3, it will imply that $k = \overline{k}$. To do this, we use (48) and (73) to write

$$U(z, \psi(x+z)) = A(z)\left[h_1(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}z} h_2(\psi(x+z)\right],$$

in the form

$$U(z, \psi(x+z)) = \frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2 \left(\frac{1}{\mu}\right)^2 f(x)\Delta(z)\frac{1+e^{-\beta z}}{1+e^{-\beta(x+z)}}, \tag{76}$$

with

$$\begin{aligned} f(x) &= e^{(\gamma-1)\beta x}(1 - e^{(1-2\gamma)\beta x}), \\ \Delta(z) &= h_1'(k(z))h_2(\psi(z)) - h_2'(k(z))h_1(\psi(z)). \end{aligned}$$

Observing that $k(z) = \psi((\phi(k(z)) - z) + z)$ and using relations (44), (45), (48), we obtain the following asymptotics

$$\begin{aligned} h_1'(k(z))h_2(\psi(z)) &\sim_{z=\infty} 2\mu(\gamma-1)e^{\gamma\beta(\phi(k(z))-z)}, \\ h_2'(k(z))h_1(\psi(z)) &\sim_{z=\infty} -2\mu\gamma e^{(1-\gamma)\beta(\phi(k(z))-z)}, \end{aligned}$$

yielding

$$\lim_{z \longrightarrow \infty} \Delta(z) = \lim_{z \longrightarrow \infty} 2\mu\left((\gamma-1)e^{\gamma\beta(\phi(k(z))-z)} - \gamma e^{(1-\gamma)\beta(\phi(k(z))-z)}\right). \tag{77}$$

Using (36), we observe that

$$\overline{V}_\mu(x) = f(x)\frac{e^{\beta\gamma\overline{x}_\mu}}{\beta(\gamma + (\gamma-1)e^{(2\gamma-1)\beta\overline{x}_\mu})}. \tag{78}$$

It then follows from (76), (77), (78) that $\lim_{z \longrightarrow \infty} U(z, \psi(x+z)) = \overline{V}_\mu(x)$ is equivalent to $\lim_{z \longrightarrow \infty} \phi(k(z)) - z = \overline{x}_\mu$.

Thus, a smooth solution (U,k) to (63)-(67), if it exists, must satisfy (72), (73) where $k=\overline{k}$ is uniquely defined in Lemma 3. Conversely, a direct computation shows that the function defined by (72), (73) with $k=\overline{k}$ is a smooth solution to (63)-(67). Finally, posing $z=\phi(y)-x$ in (72) and (73) leads to (61) and (62) where $\overline{b}$ is uniquely defined in Lemma 3. Thus, the couple $(V,\overline{b})$ defined in Proposition 6 is the unique solution to the system (50)-(54). Observe that the uniqueness of the function V comes from the uniqueness of the function $\overline{b}$ which follows from condition (67). □

To prove that $V=\overline{V}$, we proceed in two steps. First, we show in Proposition 7 that the function V solution to (63)-(67) satisfies the assumptions of the verification Lemma 2, which implies that $\overline{V}\leq V$. Second, we construct an admissible policy for problem (49), the value of which coincides with V. This latter result implies that $V\leq\overline{V}$.

Proposition 7 *The function V defined in Proposition 6 satisfies the assumptions of Lemma 2.*

Proof of Proposition 7. It is clear from (61) that V is twice continuously differentiable on any open set in $(0,\infty)\times(-\mu,\mu)$ away from the set $\{(x,y),\ x=\overline{b}(y)\}$. By construction V_x and V_{xx} are continuous across the boundary $\overline{b}$. Therefore, to prove that V is twice differentiable on $(0,\infty)\times(-\mu,\mu)$, we only have to show that the functions V_y and V_{yy} are continuous across the boundary $\overline{b}$, that is

$$\begin{aligned}\lim_{x\longrightarrow\overline{b}(y)^-} V_y(x,y) &= -\overline{b}'(y)+\nu'(y), \\ \lim_{x\longrightarrow\overline{b}(y)^-} V_{yy}(x,y) &= -\overline{b}''(y)+\nu''(y),\end{aligned} \tag{79}$$

where the function ν is defined on $(-\mu,\mu)$ by the relation

$$\nu(y)=A(\phi(y)-\overline{b}(y))\left(h_1(y)-e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-\overline{b}(y))}h_2(y)\right).$$

Let us define

$$H(y)\equiv e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-\overline{b}(y))}=\frac{h_1(\psi(\phi(y)-\overline{b}(y))}{h_2(\psi(\phi(y)-\overline{b}(y))}.$$

Using the relations (44),(45),(46),(47), we obtain

$$H'(y)=\frac{2\mu^2}{\sigma^2 y^*}(\phi'(y)-\overline{b}'(y))H(y). \tag{80}$$

Remembering the relations (69) and (71) and the definition of $\overline{k}$, we observe that

$$A(z)=-\frac{\sigma^2 y^*}{2\mu^2}\left(A'(\phi(y)-\overline{b}(y))+B'(\phi(y)-\overline{b}(y))\frac{1}{H(y)}\right).$$

We are in a position to compute the derivative of ν. We have,

$$\begin{aligned}\nu'(y) &= A(\phi(y)-\overline{b}(y))(h_1'(y)-H(y)h_2'(y))\\&+ (\phi'(y)-\overline{b}'(y))A'(\phi(y)-\overline{b}(y))(h_1(y)-H(y)h_2(y))-A(\phi(y)-\overline{b}(y))H'(y)h_2(y).\end{aligned}$$

Using the relations (80), (71), (69) and the definition of $\overline{k}$, the second term of the right-hand side is equal to

$$(\phi'(y)-\overline{b}'(y))\frac{y^*}{2\mu^2}(h_2'(y)h_1(y)-h_1'(y)h_2(y)).$$

We note that the change of variable $V(x,y)=U(\phi(y)-x,y)$ leads to the relations

$$V_y(x,y)=\phi'(y)U_z(\phi(y)-x,y)+U_y(\phi(y)-x,y), \tag{81}$$

$$\begin{aligned}V_{yy}(x,y)=\phi''(y)U_z(\phi(y)-x,y)+\phi'(y)^2U_{zz}(\phi(y)-x,y)\\+2\phi'(y)U_{zy}(\phi(y)-x,y)+U_{yy}(\phi(y)-x,y).\end{aligned} \tag{82}$$

As a consequence,

$$\begin{aligned}\nu'(y) &= (\phi'(y)-\overline{b}'(y))\frac{y^*}{2\mu^2}((h_2'(y)h_1(y)-h_1'(y)h_2(y))\\&\quad+A(\phi(y)-\overline{b}(y))(h_1'(y)-e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-\overline{b}(y))}h_2'(y))\\&= -\phi'(y)+\overline{b}'(y)+U_y(\phi(y)-\overline{b}(y),y)\\&= \overline{b}'(y)+\lim_{x\longrightarrow\overline{b}(y)^-}V_y(x,y),\end{aligned}$$

where the first equality comes from (69) and the last equality comes from (81) and from the relation $U_z(\phi(y)-\overline{b}(y),y)=-V_x(\overline{b}(y),y)=-1$. Thus (79) is satisfied. Moreover,

$$\begin{aligned}\nu''(y) &= -\phi''(y)+\overline{b}''(y)+A((\phi(y)-\overline{b}(y))h_1''(y)+B((\phi(y)-\overline{b}(y))h_2''(y)\\&\quad+(\phi'(y)-\overline{b}'(y))A'(\phi(y)-\overline{b}(y))(h_1'(y)-e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-\overline{b}(y))}h_2'(y))\\&= -\phi''(y)+\overline{b}''(y)+\frac{2r\sigma^2}{(\mu^2-y^2)^2}U((\phi(y)-\overline{b}(y),y)\\&\quad+(\phi'(y)-\overline{b}'(y))U_{zy}((\phi(y)-\overline{b}(y),y)\\&= -\phi''(y)+\overline{b}''(y)+\frac{2r\sigma^2}{(\mu^2-y^2)^2}U((\phi(y)-\overline{b}(y),y)\\&\quad=\overline{b}''(y)+\lim_{x\longrightarrow\overline{b}(y)^-}V_{yy}(x,y),\end{aligned}$$

where the last equality comes from (82) and from the relations $U_{zz}(\phi(y)-\overline{b}(y),y)=U_{zy}(\phi(y)-\overline{b}(y),y)=0$, $U_z(\phi(y)-\overline{b}(y),y)=-1$ and $U_{yy}(\phi(y)-\overline{b}(y),y)=\frac{2r\sigma^2}{(\mu^2-y^2)^2}U((\phi(y)-\overline{b}(y),y)$. Therefore, V is twice differentiable on $(0,\infty)\times(-\mu,\mu)$.

We show below that the function V has bounded first derivatives V_x and V_y. This amounts to show that $\lim_{x\longrightarrow\infty} V_x(x,y) < \infty$ and $\lim_{y\longrightarrow\mu} V_y(x,y) < \infty$. We write the change of variable $x = \phi(y) - z$ in the form $y = \psi(x+z)$ and define

$$W(x,z) \equiv V(x,\psi(x+z)) = U(z,\psi(x+z)).$$

We have

$$\begin{aligned} V_x(x,y) &= W_x(x,\phi(y)-x) - W_z(x,\phi(y)-x), \\ V_y(x,y) &= \phi'(y)W_z(x,\phi(y)-x) = \frac{\sigma^2}{\mu^2-y^2}W_z(x,\phi(y)-x). \end{aligned}$$

We then deduce that V_x is bounded if and only if

$$\lim_{z\longrightarrow\infty} W_x(x,z) - W_z(x,z) < \infty. \tag{83}$$

From (47), we deduce that V_y is bounded if and only if $\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)} W_z(x,z) < \infty$, or, equivalently if and only if

$$\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)} \Delta'(z) < \infty, \tag{84}$$

where the latter expression follows from a computation that uses (76). Using (74) we obtain that (84) is equivalent to

$$\begin{aligned} &\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)} \frac{2\mu^2}{\sigma^2 y^*}(h_1'(\overline{k}(z))h_2(\psi(z)) + h_2'(\overline{k}(z))h_1(\psi(z))) \\ &\quad + \frac{\psi'(z)}{\psi'(x+z)}(h_1'(\overline{k}(z))h_2'(\psi(z)) - h_2'(\overline{k}(z))h_1'(\psi(z))) < \infty. \end{aligned} \tag{85}$$

Recalling that $\lim_{z\longrightarrow\infty} \phi(\overline{k}(z)) - z = \overline{x}_\mu$ and observing that $\overline{k}(z) = \psi((\phi(\overline{k}(z)) - z)$, formulae (44)-(48) lead to the relations

$$\begin{aligned} \frac{1}{\psi'(x+z)} \frac{2\mu^2}{\sigma^2 y^*} h_1'(\overline{k}(z))h_2(\psi(z)) &\sim_{z=\infty} \frac{\mu}{y^*}(\gamma-1)e^{\beta x}e^{\gamma\beta\overline{x}_\mu}e^{\beta z}, \\ \frac{1}{\psi'(x+z)} \frac{2\mu^2}{\sigma^2 y^*} h_2'(\overline{k}(z))h_1(\psi(z)) &\sim_{z=\infty} -\frac{\mu}{y^*}\gamma e^{\beta x}e^{(1-\gamma)\beta\overline{x}_\mu}e^{\beta z}, \\ \frac{\psi'(z)}{\psi'(x+z)} h_1'(\overline{k}(z))h_2'(\psi(z)) &\sim_{z=\infty} -\gamma(\gamma-1)e^{\beta x}e^{\gamma\beta\overline{x}_\mu}e^{\beta z}, \\ \frac{\psi'(z)}{\psi'(x+z)} h_2'(\overline{k}(z))h_1'(\psi(z)) &\sim_{z=\infty} -\gamma(\gamma-1)e^{\beta x}e^{(1-\gamma)\beta\overline{x}_\mu}e^{\beta z}. \end{aligned}$$

Aggregating these relations in (85), we obtain that

$$\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)} W_z(x,z) = \lim_{z\longrightarrow\infty} e^{\beta x}e^{\beta z}((\gamma-1)^2 e^{\gamma\beta\overline{x}_\mu} - \gamma^2 e^{(1-\gamma)\beta\overline{x}_\mu}) = 0, \tag{86}$$

where the last equality follows from (37). Thus, V_y is bounded. We deduce from (86) that $\lim_{z\longrightarrow\infty} W_z(x,z)=0$ and from (76) that

$$W_x(x,z)\sim_{z=\infty}\frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2 f'(x)\Delta(z).$$

Since $\lim_{z\longrightarrow\infty}\phi(\overline{k}(z))-z=\overline{x}_\mu$, we obtain from (77) that (83) is satisfied. Thus V_x is bounded. Finally, we prove below that

$$\max(\mathcal{L}V-rV,1-V_x)\leq 0 \text{ on } [0,\infty)\times(-\mu,\mu).$$

Note that, by construction, $V(0,y)=0$ for all $y\in(-\mu,\mu)$ and that $\mathcal{L}V-rV=0$ on the set $\{x\leq\overline{b}(y)\}$. We first show that the mapping $x\longrightarrow V(x,y)$ is concave on the set $\{x<\overline{b}(y)\}$. The change of variable (17) and relation (63) yield that, on the set $\{x\leq\overline{b}(y)\}$,

$$V(x,y)=A(\phi(y)-x)h_1(y)+B(\phi(y)-x)h_2(y).$$

Using (69), we obtain that

$$V_{xx}(x,y)=\overline{k}'(\phi(y)-x)\frac{y^*}{2\mu^2}(h_2''(\overline{k}(\phi(y)-x)h_1(y)-h_1''(\overline{k}(\phi(y)-x)h_2(y)). \tag{87}$$

From (45), the right hand side of (87) has the same sign than

$$\overline{k}'(\phi(y)-x)\frac{y^*}{2\mu^2}(h_2(\overline{k}(\phi(y)-x)h_1(y)-h_1(\overline{k}(\phi(y)-x)h_2(y)). \tag{88}$$

Since h_1 is positive decreasing and h_2 is positive increasing, the function $(h_2(\overline{k}(\phi(y)-x)h_1(y)-h_1(\overline{k}(\phi(y)-x)h_2(y))$ is positive if and only if $\overline{k}(\phi(y)-x)>y$. Since $\overline{k}$ is increasing, this latter inequality is equivalent to $\phi(y)-x>\overline{k}^{-1}(y)$, that is $x<\overline{b}(y)$. Thus, (88) is negative since $y^*<0$. Therefore, the mapping $x\longrightarrow V(x,y)$ is concave on the set $\{x<\overline{b}(y)\}$. Because $V_x(x,y)=1$ for all $x\geq\overline{b}(y)$, we conclude that $x\longrightarrow V(x,y)$ is concave over $[0,\infty)$, and in turn that $V_x\geq 1$ on $[0,\infty)$.

It remains to show that $\mathcal{L}V-rV<0$ on the set $\{x>\overline{b}(y)\}$. On the set $\{x>\overline{b}(y)\}$, we have that $V(x,y)=x-\overline{b}(y)+\nu(y)$ and $V_{xy}(\overline{b}(y),y)=0$. We deduce the equalities $V_y(x,y)=V_y(\overline{b}(y),y)$ and $V_{yy}(x,y)=V_{yy}(\overline{b}(y),y)$. Therefore, using the fact that V is twice differentiable across $\overline{b}$, we obtain that, on the set $\{x>\overline{b}(y)\}$,

$$\begin{aligned}(\mathcal{A}V-rV)(x,y) &= \frac{1}{2\sigma^2}(\mu^2-y^2)^2V_{yy}(\overline{b}(y),y)+y-rV(x,y))\\ &= -r(x-\overline{b}(y))\\ &< 0.\end{aligned}$$

The proof of Proposition 7 is complete and thus $\overline{V} \leq V$. □

Finally, we show that the solution V can be reached by an admissible policy. Our guess is that the optimal cash reserve process is reflected along the free boundary function $\bar{b}$ on a horizontal direction in the (x, y)-plane. We formalize this using a 2-dimensional version to Skorohod's lemma established by Burdzy and Toby (1995). Specifically, there exists a unique continuous process $\{L = (L_t)_t;\, t \geq 0\}$ defined on $(\Omega, \mathcal{F}^R, \mathbb{P})$ such that, for $\mathbb{P}$-*a.e.* $\omega \in \Omega$,

- $$(\phi(Y_t(\omega)) - \phi(y) + x - L_t(\omega), Y_t(\omega)) \in [0, \bar{b}(y)] \times [\overline{y}^{**}, \mu), \quad \forall t \in [0, \tau_0], \tag{89}$$
  where $\tau_0 = \inf\{t \geq 0 \,|\, \phi(Y_t) - \phi(y) + x - L_t = 0\}$,
- $$L_0(\omega) = 0, \text{ and } t \longrightarrow L_t(\omega) \text{ is nondecreasing on } \{t \geq 0 \,:\, \phi(Y_t) - \phi(y) + x - L_t = \bar{b}(Y_t)\}, \tag{90}$$
- $$t \longrightarrow L_t(\omega) \text{ is constant on } \{t \geq 0 \,:\, (\phi(Y_t(\omega)) - \phi(y) + x - L_t(\omega), Y_t(\omega)) \in (0, \bar{b}(y)) \times (\overline{y}^{**}, \mu)\}. \tag{91}$$

Conditions (89)-(91) ensure that the policy L is admissible and that the process $\phi(Y_t) - \phi(y) + x - L_t$ is reflected in a horizontal direction whenever it hits $\bar{b}(Y_t)$.

Proposition 8 *The function V can be attained by an admissible policy and thus $V \leq \overline{V}$.*

Proof of Proposition 8 Let us consider the process $\overline{D} = \{\overline{D}_t; t \geq 0\}$ with

$$\overline{D}_t = \left((x - \bar{b}(y))^+ \mathbb{1}_{y \geq \overline{y}^{**}} + x \mathbb{1}_{y \leq \overline{y}^{**}}\right) \mathbb{1}_{t=0} + L_t \mathbb{1}_{t>0}, \tag{92}$$

where L is defined by (89)-(91) and let us consider the continuous process

$$X_t \equiv \phi(Y_t) - \phi(y) + x - \overline{D}_t.$$

A computation based on Itô's formula yields that, for all $t \geq 0$,

$$\begin{aligned} \mathbb{E}\left[e^{-rt\wedge\tau_0} V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0})\right] &= V(x, y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs} V_x(X_s, Y_s)\, d\overline{D}_s\right] \\ &= V(x, y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\, d\overline{D}_s\right] \end{aligned} \tag{93}$$

where the second equality comes from (90) and (91) along with the fact that $V_x(\bar{b}(y), y) = 1$. Letting t go to ∞ in (93) yields

$$V(x, y) = \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\, d\overline{D}_s\right] \leq \overline{V}.$$

$\square$

Thus, from Propositions 7 and 8, the function V defined in Proposition 6 coincides with the value function $\overline{V}$ of problem (49). Equation (92) provides the optimal dividend policy: The function $\overline{b}$ corresponds to the dividend boundary function of the shareholders' problem (49). The optimal cash reserve process is reflected along the function $\overline{b}$ on a horizontal direction in the (x, y)-plane.

8.3.2 Equity issuance

We are now ready to prove Theorem 1 and to solve problem (8):

$$V^*(x, y) = \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dD_t - dI_t)\right].$$

We proceed as in the previous section where equity issuance was not allowed. The verification Lemma 2 has to be adapted in the following way.

Lemma 4 *(Verification Lemma) Assume there exists a function V defined on $[0, \infty) \times (-\mu, \mu)$ that satisfies*

1. *V is twice differentiable almost everywhere,*
2. *V has bounded first derivatives,*
3. *$\max(-V(0, y), V_x(0, y) - p) = 0$ for all $y \in (-\mu, \mu)$ and,*

$$\max(\mathcal{A}V - rV, 1 - V_x, V_x - p) \leq 0 \text{ almost everywhere on } [0, \infty) \times (-\mu, \mu),$$

then $V \geq V^*$.

Proof of Lemma 4. See the online appendix.

We first assume that the proportional issuance costs p satisfies $p \geq \overline{p} = \overline{V}'_\mu(0)$. We prove that the firm value V^* solution to (8) coincides with $\overline{V}$ solution to (49), so that also the functions b^* and $\overline{b}$ coincide. This proves Theorem 1 when $p \geq \overline{p}$. The result is a direct consequence of the following Lemma.

Lemma 5 *Let us assume that $p \geq \overline{p}$, then $\overline{p} > \overline{V}_x(x, y)$ for all $(x, y) \in [0, \infty) \times \in (-\mu, \mu)$.*

Proof of Lemma 5. See the online appendix.

From Lemma 5, $\overline{V}$ satisfies the assumptions of Lemma 4. It follows that $V^*(x,y) \leq \overline{V}(x,y)$. On the other hand, considering the policies $I^* = 0$ and D^* defined in (92) lead to the inequality $V^*(x,y) \geq \overline{V}(x,y)$, thus the result.

The case $p \leq \overline{p}$ is much more involved. The analysis relies on the following technical Proposition.

Proposition 9 *The following holds.*

(i) Fix $z_i > z^* = \phi(y^*)$, the relation

$$-h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z)) + \frac{2\mu^2}{y^*}p = 0 \tag{94}$$

uniquely defines over $[z_i, \infty)$ a continuously differentiable increasing function $k > \psi$ that satisfies

$$\lim_{z \longrightarrow \infty} \phi(k(z)) - z = x_\mu, \tag{95}$$

$$k(z) \underset{+\infty}{\sim} \psi(z + x_\mu). \tag{96}$$

(ii) The equation

$$\frac{p}{\psi'(z)} h_1(\psi(z)) + \int_z^\infty h_1'(k(u))\, du = 0 \tag{97}$$

has a unique solution $z_i^* \in (z^*, \infty)$.

(iii) Let us denote $y_i^* = \psi(z_i^*)$ and let us consider the function k^* with

$$k^*(z) = k_1(z)\mathbb{1}_{z \leq z_i^*} + k_2(z)\mathbb{1}_{z \leq z_i^*}, \tag{98}$$

where (94) characterizes k_2, and where k_1 is the solution to the ordinary differential equation $k_1'(z) = \Theta(k_1(z), z)$ with terminal condition $k_1(z_i^*) = k_2(z_i^*)$ where Θ is defined by (60). Then, k^* is a well defined continuous increasing function over (z^*, ∞) and continuously differentiable over $\mathbb{R} \setminus \{z_i^*\}$.

Proof of Proposition 9. See the online appendix. □

We state now the main result of this section.

Proposition 10 *Let us consider the function k^* defined by (98) and the function b^* defined by the relation*

$$b^*(y) = \max(0, b_1(y))\mathbb{1}_{y \leq y_i^*} + b_2(y)\mathbb{1}_{y \geq y_i^*}, \tag{99}$$

with $b_2(y) = (\phi - k_2^{-1})(y)$ for any $y \geq y_i^*$ and, $b_1(y) = (\phi - k_1^{-1})(y)$ for any $y \leq y_i^*$. The function b^* is well defined and positive on $[y^{**}, \mu)$ with $y^{**} = b^{-1}(0)$. It is differentiable on $(y^{**}, \mu) \setminus \{y^{i*}\}$ where $y^{i*} \equiv k_1(z_i^*) = k_2(z_i^*)$. Moreover, let us consider the function $(x, y) \longrightarrow V(x, y)$ defined on $\mathbb{R}^+ \times (-\mu, \mu)$ by the relations

- *For $y \in [y_i^*, \mu)$,*

$$\begin{cases} V(x,y) = A\,(\phi(y) - x)\,h_1(y) + B\,(\phi(y) - x)\,h_2(y), & \forall\, 0 \leq x \leq b^*(y), \\ V(x,y) = x - b^*(y) + V(b^*(y), y), & \forall\, x \geq b^*(y), \end{cases}$$

where for $z > z_i^*$:

$$\begin{cases} A(z) = A(z_i^*) + \int_{z_i^*}^{z} \frac{y^*}{2\mu^2} h_2'(k^*(u))\,du, \\ B(z) = B(z_i^*) - \int_{z_i^*}^{z} \frac{y^*}{2\mu^2} h_1'(k^*(u))\,du \end{cases} \tag{100}$$

and,

$$\begin{cases} A(z_i^*) = \frac{p}{\psi'(z_i^*)} \frac{y^*}{2\mu^2} h_2(\psi(z_i^*)), \\ B(z_i^*) = -\frac{p}{\psi'(z_i^*)} \frac{y^*}{2\mu^2} h_1(\psi(z_i^*)). \end{cases} \tag{101}$$

- *For $y \in (-\mu, y_i^*]$,*

$$\begin{cases} V(0,y) = 0, & \forall -\mu < y < \underline{y}, \\ V(x,y) = A\,(\phi(y) - x)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{y^*}(\phi(y)-x)} h_2(y)\right), & \forall\, 0 \leq x \leq b^*(y), \\ V(x,y) = x - b^*(y) + V(b^*(y), y), & \forall\, x \geq b^*(y) \end{cases}$$

where for $z \leq z_i$,

$$A(z) = \frac{\sigma^2}{4}\left(\frac{y^*}{\mu}\right)^2 \left(\frac{1}{\mu}\right)^2 \left(h_1'(k^*(z)) e^{-\frac{2}{\sigma^2}\frac{\mu^2}{y^{*}} z} - h_2'(k^*(z))\right). \tag{102}$$

Then, the triple (V, y_i^*, b^*) is the unique solution to the system (11)-(16).

Proof of Proposition 10. The proof follows the same route than the proof of Proposition

6. We first consider a solution (U, z_i, k) to the system

$$\begin{aligned}
\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z,y) - rU(z,y) &= 0 \quad \text{on } \{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}, && (103)\\
U(z,\psi(z)) &= 0 \quad \forall z \leq z_i, && (104)\\
U_z(z,\psi(z)) &= -p \quad \forall z \geq z_i, && (105)\\
U_z(z,y) &= -1, \text{ for } k(z) \leq y, && (106)\\
U_{xy}(z,k(z)) &= 0, && (107)\\
\lim_{z \longrightarrow \infty} U(z,\psi(x+z)) &= V_\mu(x). && (108)
\end{aligned}$$

The relations derived in the proof of Proposition 6 hold true for $z \leq z_i$, so that the solution (U, z_i, k) satisfies (72), (73) and (75) for $z \leq z_i$.

Note that (68) and (69) hold true for any $(z,y) \in \{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}$. We then deduce from (105) that (94) characterizes the function k on $[z_i, \infty)$.

Consider (104) and take the derivative with respect to z of $U(z,\psi(z))$. One get

$$U_z(z,\psi(z)) + \psi'(z) U_y(z,\psi(z)) = 0. \tag{109}$$

Then, using (68) and (105), Equations (104) and (109) evaluated at z_i yield

$$\begin{cases} A(z_i) = \frac{p}{\psi'(z_i)} \frac{y^*}{2\mu^2} h_2(\psi(z_i)), \\ B(z_i) = -\frac{p}{\psi'(z_i)} \frac{y^*}{2\mu^2} h_1(\psi(z_i)). \end{cases} \tag{110}$$

We then obtain from (69) that

$$\begin{cases} A(z) = A(z_i) + \int_{z_i}^{z} \frac{y^*}{2\mu^2} h_2'(k(u))\, du, \\ B(z) = B(z_i) - \int_{z_i}^{z} \frac{y^*}{2\mu^2} h_1'(k(u))\, du. \end{cases} \tag{111}$$

Thus, a smooth solution (U, z_i, k) to (103)-(107) satisfies $U(z,y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}$ and the relations (73), (75), (110), (111). We prove below that such a smooth solution satisfies (108) if and only if $z_i = z_i^*$. This will imply that $k = k^*$. We have

$$U(z,\psi(x+z)) = A(z)h_1(\psi(x+z)) + B(z)h_2(\psi(x+z)) \tag{112}$$

where (70) and (73) define A and B for $z \leq z_i$ and (110) and (111) define A and B for $z \geq z_i$. We deduce from (45), (48) and (96) that

$$h_1'(k(u)) \underset{+\infty}{\sim} (\gamma - 1)e^{\gamma\beta(u+x_\mu)} \quad \text{and} \quad h_2'(k(u)) \underset{+\infty}{\sim} -\gamma e^{(1-\gamma)\beta(u+x_\mu)}. \tag{113}$$

It follows that

$$\int_{z_i}^{\infty} \frac{y^*}{2\mu^2} h_2'(k(u))du = -\infty, \quad \int_{z_i}^{\infty} \frac{y^*}{2\mu^2} h_1'(k(u))du < \infty, \tag{114}$$

and also that,

$$\int_{z_i}^{z} h_2'(k(u))\, du \underset{+\infty}{\sim} \int_{z_i}^{z} -\gamma e^{(1-\gamma)\beta(z+x_\mu)}\, du, \tag{115}$$

yielding

$$\begin{aligned} \int_{z_i}^{z} -\gamma e^{(1-\gamma)\beta(u+x_\mu)}\, du \;&=\; \frac{\gamma}{\gamma-1}\frac{1}{\beta} e^{-(\gamma-1)\beta(z+x_\mu)} \\ &\quad -\frac{\gamma}{\gamma-1}\frac{1}{\beta} e^{-(\gamma-1)\beta(z_i+x_\mu)}. \end{aligned}$$

Using (48), we deduce that

$$\begin{aligned} &\lim_{z\longrightarrow\infty} \int_{z_i}^{z} \frac{y^*}{2\mu^2} h_2'(k(u))du\; h_1(\psi(x+z)) \\ =\;& \frac{\gamma}{\beta(\gamma-1)}\frac{y^*}{\mu^2} e^{-(\gamma-1)\beta x_\mu} e^{(\gamma-1)\beta(x+z)} - \lim_{z\longrightarrow\infty} \frac{\gamma}{\beta(\gamma-1)}\frac{y^*}{\mu} e^{-(\gamma-1)\beta(z_i+x_\mu)} e^{(\gamma-1)\beta(x+z)} \\ =\;& \frac{\gamma}{\beta(\gamma-1)}\frac{y^*}{\mu^2} e^{-(\gamma-1)\beta x_\mu} e^{(\gamma-1)\beta(x+z)}, \end{aligned} \tag{116}$$

where the last equality comes from the fact that $\gamma < 0$. We also have,

$$\int_{z_i}^{z} \frac{y^*}{2\mu^2} h_1'(k(u))\, du \;=\; \int_{z_i}^{\infty} \frac{y^*}{2\mu^2} h_1'(k(u))\, du - \int_{z}^{\infty} \frac{y^*}{2\mu^2} h_1'(k(u))\, du$$

from which we deduce that,

$$\begin{aligned} \lim_{z\longrightarrow\infty} \int_{z_i}^{z} \frac{y^*}{2\mu^2} h_1'(k(u))\, du\; h_2(\psi(x+z)) \;&=\; \Big(\lim_{z\longrightarrow\infty} \int_{z_i}^{\infty} \frac{y^*}{\mu} h_1'(k(u))\, du\; e^{-\beta\gamma(x+z)} \\ &\quad - \int_{z}^{\infty} \frac{y^*}{\mu} e^{\beta\gamma(u+x_\mu)}(\gamma-1)\, du\; e^{-\beta\gamma(x+z)} \Big). \end{aligned}$$

Thus,

$$\lim_{z\longrightarrow\infty} \int_{z_i}^{z} \frac{y^*}{2\mu^2} h_1'(k(u))\, du\, h_2(\psi(x+z)) = -\frac{1-\gamma}{\beta\gamma}\frac{y^*}{\mu} e^{\beta\gamma x_\mu} + \lim_{z\longrightarrow\infty} e^{-\beta\gamma(x+z)} \int_{z_i}^{\infty} \frac{y^*}{\mu} h_1'(k(u))\, du \tag{117}$$

Using (112), (110) and (111) together with (40), (116), (117) we obtain

$$\begin{aligned} \lim_{z\longrightarrow\infty} U(z,\psi(x+z)) \;&=\; \lim_{z\longrightarrow\infty} \Big(\int_{z_i}^{z} \frac{y^*}{2\mu^2} h_2'(k(u))\, du\; h_1(\psi(x+z)) + B(z_i) h_2(\psi(x+z)) \\ &\quad - \int_{z_i}^{z} \frac{y^*}{2\mu^2} h_1'(k(u))\, du\; h_2(\psi(x+z)) \Big) \\ &=\; V_\mu(x) + \lim_{z\longrightarrow\infty} \left(B(z_i) h_2(\psi(x+z)) - \int_{z_i}^{\infty} \frac{y^*}{2\mu^2} h_1'(k(u))\, du\; 2\mu e^{-\beta\gamma(x+z)} \right) \\ &=\; V_\mu(x) - \lim_{z\longrightarrow\infty} \left(\frac{p}{\psi'(z_i)} h_1(\psi(z_i)) + \int_{z_i}^{\infty} h_1'(k(u))\, du \right) \frac{y^*}{\mu} e^{-\beta\gamma(x+z)}. \end{aligned}$$

Since $\gamma < 0$, the function U satisfies (108) if and only if z_i satisfies (97), or equivalently if $z_i = z_i^*$. Therefore, a smooth solution (U, z_i, k) to (103)-(108) satisfies $U(z,y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}$ and the relations (73), (75), (110), (111) with $k = k^*$ (and thus $z_i = z_i^*$). Conversely, a computation shows that $U(z,y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}$ and the relations (73), (75), (110), (111) with $k = k^*$ is a solution to the system (103)-(108).

Finally, from Proposition 9, k_1 and k_2 in (98) are increasing so that, b^* in (99) is indeed a well defined function which is not differentiable at y^{i*} from assertion (iii) of Proposition 9. Then, using the change of variable $z = \phi(y) - x$, we obtain that (V, y_i^*, b^*) is a solution to the system (11)-(16). Observe that the uniqueness of the function V comes from the uniqueness of the function b^* that follows from condition (108). □

Proceeding as in the previous section, we show the following

Proposition 11 *The function V defined in Proposition 10 satisfies the assumptions of Lemma 4 and thus $V \geq V^*$.*

Proof of Proposition 11. The proof relies on arguments developed in Proposition 7. We first show that the function V is $\mathcal{C}^1$ on the domain $(0,\infty) \times (-\mu,\mu)$ and is $\mathcal{C}^2$ on the domain $(0,\infty) \times (-\mu,\mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. By construction, V is twice continuously differentiable on any open set in $(0,\infty) \times (-\mu,\mu)$ away from the set $\{(x,y),\, x = b^*(y)\}$. Since b^* is differentiable on $(y^{**},\mu) \setminus \{y^{i*}\}$, we can proceed as in the proof of Proposition 7 to prove that V is of class $\mathcal{C}^2$ on $[0,\infty) \times (-\mu,\mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. Also by construction, the study of the $\mathcal{C}^1$-differentiability of V^* does not involve the derivative of b^*: to establish that V is $\mathcal{C}^1$ on $(0,\infty) \times (-\mu,\mu)$, it is sufficient to check that A and B are continuously differentiable at z_i^*. For the continuity of A (or equivalently of B) at z_i^*, observe that (73) and (94) evaluated at z_i^*, lead to (101). The differentiability of A and B at z_i^* comes from (69) and (100). Thus, V is twice differentiable almost everywhere.

Second, let us fix any $y \in (y^{**},\mu)$. We deduce from the proof of Proposition 7 that the mapping $x \to V(x,y)$ is concave on $[0,\infty) \setminus \{\phi(y) - z_i^*\}$ if and only if

$$(k^*)'(\phi(y)-x)(h_2(k^*(\phi(y)-x)h_1(y) - h_1(k^*(\phi(y)-x)h_2(y)) < 0. \tag{118}$$

Note that (118) is well defined since $\phi(y) - x \neq z_i^*$ on the considered domain, such that the derivative of k^* is well defined. The reasoning developed in the proof of Proposition 7 shows that (118) holds true. Now, because V is linear in x outside $\{x < b^*(y)\}$ and that $V_x(b^*(y),y) = 1$ for any $y \in (-\mu,\mu)$, we deduce from the concavity of $x \to V(x,y)$ that

$V_x(x,y) \geq 1$ on $[0,\infty) \setminus \{\phi(y) - z_i^*\}$. Then, because V is $\mathcal{C}^1$ on $(0,\infty) \times (-\mu,\mu)$, one obtains that $V_x(x,y) \geq 1$ on $(0,\infty) \times (-\mu,\mu)$.

Finally, the concavity and $\mathcal{C}^1$-differentiability properties together with the fact that $V_x(0,y) \leq p$ for any $y \in (-\mu,\mu)$ lead to $V_x(x,y) \leq p$ on $(0,\infty) \times (-\mu,\mu)$.

The arguments developed in the proof of Proposition 7 show that $\mathcal{L}V - rV \leq 0$ on $(0,\infty) \times (-\mu,\mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. Aggregating all these results, we obtain that, almost everywhere on $(0,\infty) \times (-\mu,\mu)$,

$$\max(\mathcal{A}V - rV, 1 - V_x, V_x - p) \leq 0.$$

Observe also that V satisfies by construction $\max(-V(0,y), V_x(0,y) - p) = 0$ for all $y \in (-\mu,\mu)$.

To conclude, it remains to show that the function V has bounded first derivatives. We have $V_x(x,y) \leq p$ on $(0,\infty) \times (-\mu,\mu)$ such that $(x,y) \longrightarrow V_x(x,y)$ is bounded over $[0,\infty) \times (-\mu,\mu)$. From the expression of V in Proposition 10, we deduce that V_y is bounded if and only if

$$\lim_{y \longrightarrow \mu} V_y(x,y) < \infty.$$

That is, as shown in Proposition 7, if and only if

$$\lim_{z \longrightarrow \infty} \frac{1}{\psi'(x+z)} W_z(x,z) < \infty$$

where $W(x,z) = V(x,\psi(x+z)) = U(z,\psi(x+z))$.

We have that

$$\begin{aligned}
W(x,z) &= A(z)h_1(\psi(x+z)) + B(z)h_2(\psi(x+z)) \\
&= A(z_i^*)h_1(\psi(x+z)) + B(z_i^*)h_2(\psi(x+z)) \\
&\quad + \int_{z_i^*}^{z} \frac{y^*}{2\mu^2} h_2'(k(u))\, du\, h_1(\psi(x+z)) \\
&\quad - \int_{z_i^*}^{z} \frac{y^*}{2\mu^2} h_1'(k(u))\, du\, h_2(\psi(x+z)).
\end{aligned}$$

This leads to

$$\begin{aligned}
\frac{1}{\psi'(x+z)}W_z(x,z) &= A(z_i^*)h_1'(\psi(x+z)) + B(z_i^*)h_2'(\psi(x+z)) \\
&+\frac{y^*}{2\mu^2}h_2'(k(z))\frac{h_1(\psi(x+z))}{\psi'(x+z)} - \frac{y^*}{2\mu^2}h_1'(k(z))\frac{h_2(\psi(x+z))}{\psi'(x+z)} \\
&+\int_{z_i^*}^{z}\frac{y^*}{2\mu^2}h_2'(k(u))\,du\,h_1'(\psi(x+z)) \\
&-\int_{z_i^*}^{z}\frac{y^*}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z)),
\end{aligned}$$

or, equivalently using (46), (47),

$$\begin{aligned}
\frac{1}{\psi'(x+z)}W_z(x,z) &= A(z_i^*)h_1'(\psi(x+z)) + B(z_i^*)h_2'(\psi(x+z)) \\
&+\frac{y^*\sigma^2}{2\mu^2}\frac{h_2'(k(z))}{h_2(\psi(x+z))} - \frac{y^*\sigma^2}{2\mu^2}\frac{h_1'(k(z))}{h_1(\psi(x+z))} \\
&+\int_{z_i^*}^{z}\frac{y^*}{2\mu^2}h_2'(k(u))\,du\,h_1'(\psi(x+z)) \\
&-\int_{z_i^*}^{z}\frac{y^*}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z)). \qquad (119)
\end{aligned}$$

Using (113), (114), (115) and (48), we have that

$$h_1'(\psi(x+z)) \underset{+\infty}{\sim} (\gamma-1)e^{\gamma\beta(x+z)}, \quad h_2'(\psi(x+z)) \underset{+\infty}{\sim} -\gamma e^{(1-\gamma)\beta(x+z)},$$

$$\frac{y^*\sigma^2}{2\mu^2}\frac{h_2'(k(z))}{h_2(\psi(x+z))} \underset{+\infty}{\sim} -\frac{y^*}{2\mu^2}\frac{1}{\beta}\gamma e^{(1-\gamma)\beta x_\mu}e^{\gamma\beta x}e^{\beta z}, \quad \frac{y^*\sigma^2}{2\mu^2}\frac{h_1'(k(z))}{h_1(\psi(x+z))} \underset{+\infty}{\sim} -\frac{y^*}{2\mu^2}\frac{1}{\beta}(1-\gamma)e^{\gamma\beta x_\mu}e^{(1-\gamma)\beta x}e^{\beta z},$$

$$\int_{z_i^*}^{z} h_2'(k(u))\frac{y^*}{2\mu^2}du\ h_1'(\psi(x+z)) \underset{+\infty}{\sim} \frac{y^*}{2\mu^2}\gamma\frac{1}{\beta}e^{\gamma\beta x}e^{-(\gamma-1)\beta x_\mu}e^{\beta z}.$$

and

$$\begin{aligned}
\lim_{z\longrightarrow\infty}\int_{z_i^*}^{z}\frac{y^*}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z)) &= \lim_{z\longrightarrow\infty}\Big(\int_{z_i^*}^{\infty}\frac{y^*}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z)) \\
&- \int_{z}^{\infty}\frac{y^*}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z))\Big).
\end{aligned}$$

Aggregating all these relations in (119), a last computation shows that proving $\lim_{y\longrightarrow\infty} V_y(x,y) < \infty$ is equivalent to prove

$$\lim_{z\longrightarrow\infty}\left(-B(z_i^*) + \frac{y^*}{2\mu^2}\int_{z_i^*}^{z} h_1'(k(u))\,du\right)\gamma e^{(1-\gamma)\beta(x+z)} < \infty.$$

Noting that $\beta > 0$ and using (101), we obtain that $\lim_{y \longrightarrow \infty} V_y(x,y) < \infty$ if and only if z_i^* satisfies (97), which indeed holds true by definition of z_i^* and concludes the proof of Proposition 11, from which it follows that $V^* \leq V$. □

The next proposition establishes the converse inequality

Proposition 12 *The function V can be attained by an admissible policy and thus $V \leq \overline{V}$.*

Proof of Proposition 12. The proof follows exactly the same arguments than those developed in Proposition 8. Let L^{b^*} and L^0 positive continuous increasing processes such that the process

$$\phi(Y_t) - \phi(y) + x - L_t^{b^*} + L_t^0$$

is reflected in a horizontal direction whenever $X_t = b^*(Y_t)$ and whenever $X_t = 0$. Following the results of Burdzy and Toby (1995), the processes $L_t^{b^*}$ and L_t^0 are well-defined. Then, the policies

$$D_t^* = \left((x - b^*(y))^+ \mathbb{1}_{y \geq y^{**}} + x \mathbb{1}_{y \leq y^{**}}\right) \mathbb{1}_{t=0} + L_t^{b^*} \mathbb{1}_{t>0},$$

$$I_t^* = L_t^0 \mathbb{1}_{Y_{\tau_0} > y_i^*} \mathbb{1}_{t>0},$$

are admissible and a computation based on Itô's formula yields

$$V(x,y) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dD_t^* - dI_t^*)\right]$$

which concludes the proof of Proposition 12. □

Thus, from Propositions 11 and 12, the function V defined in Proposition 10 coincides with the value function V^* of problem (8). The function b^* corresponds to the dividend boundary function of the shareholders' problem (8). The optimal cash reserve process is reflected along the function b^* on a horizontal direction in the (x,y)-plane. The threshold y_i^* corresponds to the level of the profitability prospects above which new shares are issued when cash reserves are depleted.

To conclude the proof of our main Theorem 1, it remains to show the properties of the dividend boundary function b^*. The next proposition yields these results.

Proposition 13 *The function b^* is increasing for $y \leq y^{i*}$ and decreasing in y for $y \geq y^{i*}$, where $y^{i*} = k^*(z_i^*)$. The function b^* attains its maximum at y^{i*}, $b^*(y^{i*}) > x_\mu$, and $\overline{y}^{**} > y^{**} > y^*$.*

Proof of Proposition 13. Given the previous results, we only need to show that b^* is increasing over $[y^{**}, y^{i*}]$, decreasing over $[y^{i*}, \mu)$ and that $\overline{y}^{**} > y^{**} > y^*$ to prove Proposition 13. First, we show that b^* is decreasing over $[y^{i*}, \mu)$. Let us introduce the notation $\tilde{b}(z) \equiv b^*(k^*(z)) = \phi(k^*(z)) - z$ with $z > z_i^*$. Because $(k^*)^{-1}$ is increasing, we deduce from the relation $b^*(y) = \tilde{b}((k^*)^{-1}(y))$ that, for any $z > z_i^*$, $\tilde{b}'(z) = (k^*)'(z)\phi'(k^*(z)) - 1$ has the same sign than $b^{*'}(y)$ for any $y > y^{i*}$. In addition, the relation $k^*(z) = \psi(\tilde{b}(z) + z)$ leads to $k(^*)'(z) = (1 + \tilde{b}'(z))\psi'(z + \tilde{b}(z))$, that is $\tilde{b}'(z) = \frac{k^{*'}(z)}{\psi'(z+\tilde{b}(z))} - 1$. Therefore, to show that b^* is decreasing over $[y^{i*}, \mu)$, we prove that $\frac{k^{*'}(z)}{\psi'(z+\tilde{b}(z))} - 1 > 0$ for any $z > z_i^*$. This latter inequality follows from a computation developed in Lemma 7 in the additional appendix. Then, since b^* is decreasing on $[y^{i*}, \mu)$ and $b^*(\mu) = x_\mu$, we have that $b^*(y^{i*}) > x_\mu$.

It remains to show that b^* is increasing over $[y^{**}, y^{i*}]$ and that $\overline{y}^{**} > y^{**} > y^*$. We obtain from our previous results that

$$V(x, y; b^*) = V^*(x, y) \geq \mathbb{E}\left[\int_0^{\tau_0} e^{-rt} d\overline{D}_t\right] = V(x, y; \overline{b}),$$

where $V(x, y; b^*)$ is defined in Proposition 10, and $V(x, y; \overline{b})$ is defined in Proposition 6. Equation (92) defines the process $\overline{D}$. In particular we have that

$$V(b^*(\overline{y}^{**}), \overline{y}^{**}; b^*) \geq V(\overline{b}(\overline{y}^{**}), \overline{y}^{**}; \overline{b}) = V(0, \overline{y}^{**}; \overline{b}),$$

which implies $b^*(\overline{y}^{**}) \geq \overline{b}(\overline{y}^{**})$. Let us recall that, for $y \leq y^{i*}$ the functions b^* and $\overline{b}$ satisfy (55) and (57). Then, the non crossing property of ordinary differential equations implies $b^*(y) \geq \overline{b}(y)$, for $y \leq y^{i*}$. It then follows from the proof of Lemma 3 that the function b^* is increasing for $y \leq y^{i*}$ and that $\overline{y}^{**} \geq y^{**}$. The same reasoning than in the proof of Lemma 3 shows that $y^{**} > y^*$. Finally, a computation shows that y^{i*} is increasing in the proportional issuance cost p. Recalling that $V(x, y; b^*) = V(x, y; \overline{b})$ for $p = \overline{p}$, we then deduce that $\overline{y}^{**} > y^{**}$ for $1 < p < \overline{p}$. This concludes the proof of Proposition 13. The proof of Theorem 1 is complete. $\square$

Proofs of Lemma 4 and Lemma 2. The proof of Lemma 2 follows from a straightforward adaptation of the proof of Lemma 4 that we show below.

Let us consider a pair of admissible policies D and I and let us write $D_t = D_t^c + D_t^d$, $I_t = I_t^c + I_t^d$, where D_t^c (resp. I_t^c) is the continuous part of D_t (resp. I_t) and D_t^d (resp. I_t^d) is the pure discontinuous part of D_t (resp. I_t). We recall the dynamics of the cash reserve process and of the profitability prospects process:

$$\begin{aligned} dX_t &= Y_t\,dt + \sigma dB_t - dD_t + \frac{dI_t}{p}, \\ dY_t &= \frac{\mu^2 - Y_t^2}{\sigma}\,dB_t. \end{aligned}$$

Applying the generalized Itô's formula to a function V that satisfies the assumptions of Lemma 4, we can write for $\tau_0 = \inf\{t \geq 0,\, X_t = 0\}$,

$$\begin{aligned} e^{-r(t\wedge\tau_0)}V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0}) &= V(x,y) + \int_0^{t\wedge\tau_0} e^{-rs}\left(\mathcal{L}V(X_{s-},Y_s) - rV(X_{s-},Y_s)\right)ds \\ &+ \int_0^{t\wedge\tau_0} e^{-rs}V_x(X_{s-},Y_s)\sigma dB_s \\ &+ \int_0^{t\wedge\tau_0} e^{-rs}V_y(X_{s-},Y_s)\frac{\mu^2 - Y_s^2}{\sigma}dW_s \\ &- \int_0^{(t\wedge\tau_0)} e^{-rs}V_x(X_s,Y_s)\,dD_s^c \\ &- \int_0^{(t\wedge\tau_0)} e^{-rs}V_x(X_s,Y_s)\,\frac{dI_s^c}{p} \\ &+ \sum_{s\leq t\wedge\tau_0} e^{-rs}(V(X_s,Y_s) - V(X_{s-},Y_s))(\mathbb{1}_{(\Delta X)_s>0} + \mathbb{1}_{(\Delta X)_s<0}), \end{aligned}$$

where $(\Delta X)_s = X_s - X_{s^-}$. By assumption, the second term of the right-hand side is negative and, because V has bounded first derivatives, the two stochastic integrals are centered square integrable martingales. Taking expectations and using $1 \leq V_x \leq p$, we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0})\right] &\leq V(x,y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\,dD_s^c\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\,dI_s^c\right] \\ &+ \mathbb{E}\left[\sum_{s\leq t\wedge\tau_0} e^{-rs}(V(X_s,Y_s) - V(X_{s-},Y_s)(\mathbb{1}_{(\Delta X)_s>0} + \mathbb{1}_{(\Delta X)_s<0}))\right]. \end{aligned}$$

Observe that there are two types of jumps for the cash reserve process $(X_t)_{t\geq 0}$. When there is a dividend distribution, $(\Delta X)_s < 0$ so that $X_{s-} - X_s = D_s - D_{s-} > 0$. When there is an issue of shares, $(\Delta X)_s > 0$ and $X_s - X_{s-} = I_s - I_{s-} > 0$. Therefore, the Mean-Value theorem gives the existence of a random variable $\theta \in [X_s, X_{s-}]$ when $X_{s-} - X_s = D_s - D_{s-} > 0$ and a random variable $\eta \in [X_{s-}, X_s]$ when $X_s - X_{s-} = \frac{I_s - I_{s-}}{p} > 0$ such that:

- on the set $\{X_{s-} - X_s = D_s - D_{s-}\}$, $V(X_s, Y_s) - V(X_{s-}, Y_s) = -V_x(\theta, Y_s)(X_{s-} - X_s)$,
- on the set $\{X_s - X_{s-} = \frac{I_s - I_{s-}}{p}\}$, $V(X_s, Y_s) - V(X_{s-}, Y_s) = V_x(\eta, Y_s)(X_s - X_{s-})$.

Therefore,

$$\begin{aligned}
V(X_s, Y_s) - V(X_{s-}, Y_s) &= (V(X_s, Y_s) - V(X_{s-}, Y_s))(\mathbb{1}_{(\Delta X)_s > 0} + \mathbb{1}_{(\Delta X)_s < 0}) \\
&= V_x(\eta, Y_s)(X_s - X_{s-})\mathbb{1}_{(\Delta X)_s > 0} - V_x(\theta, Y_s)(X_{s-} - X_s)\mathbb{1}_{(\Delta X)_s < 0} \\
&= V_x(\eta, Y_s)(\frac{I_s - I_{s-}}{p})\mathbb{1}_{(\Delta X)_s > 0} - V_x(\theta, Y_s)(D_s - D_{s-})\mathbb{1}_{(\Delta X)_s < 0} \\
&\leq (I_s - I_{s-})\mathbb{1}_{(\Delta X)_s > 0} - (D_s - D_{s-})\mathbb{1}_{(\Delta X)_s < 0}.
\end{aligned}$$

Finally, we obtain

$$V(x, y) \geq \mathbb{E}\left[e^{-r(t\wedge\tau_0)} V(X_{(t\wedge\tau_0)}, Y_{(t\wedge\tau_0)})\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\,(dD_s - dI_s)\right].$$

In order to get rid of the first term of the right-hand side, we observe that under the assumptions of Lemma 4, we have $V(x, y) \leq V(0, y) + px$ that implies

$$\mathbb{E}\left[e^{-r(t\wedge\tau_0)} V(X_{(t\wedge\tau_0)}, Y_{(t\wedge\tau_0)})\right] \leq e^{-rt} V_\mu(0) + p\mathbb{E}[e^{-rt} X_t].$$

Letting t go to ∞ yields

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-r(t\wedge\tau_0)} V(X_{(t\wedge\tau_0)}, Y_{(t\wedge\tau_0)})\right] = 0,$$

and thus

$$V(x, y) \geq \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,(dD_s - dI_s)\right],$$

which ends the proof of Lemma 4. □

Proof of Lemma 3. Observe that the function f defined in (57) does not satisfy the Lipschitz condition on an open domain containing (x, μ) with $x \geq 0$, so that the existence and uniqueness of a solution to (55), (56) require a specific analysis.

We remark that, the denominator of (57) is strictly positive if and only if $x > l_1(y)$ where

$$l_1(y) = \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu}).$$

Thus, f satisfies a local Lipschitz condition with respect to x in $\mathcal{D}$, where $\mathcal{D} = \{(x,y) \in \mathbb{R} \times (-\mu,\mu) \,|\, x > l_1(y)\}$. Thus, for any (x,y), there exists a unique solution $g_{x,y}$ to (55) defined on a maximal interval $I \subset (-\mu,\mu)$ passing through (x,y).

Second, the numerator of (57) is strictly positive if and only if[32] $x > l_2(y)$ where,

$$l_2(y) = \frac{y^*}{\mu}\phi\left(\frac{yy^*}{\mu - r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu}}\right).$$

The function l_2 is continuously increasing on $[-\mu,\mu]$ and satisfies the inequality $l_2(y) > l_1(y)$ for any $y \in (0,\mu]$. Furthermore, $l_1(0) = l_2(0) = 0$ and $l_1(\mu) < l_2(\mu) = \overline{x}_\mu$. To see the last equality, use (37) and remark that $2\phi(y^*) = \phi(\frac{y^*}{1-r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu^2}})$. This leads to $l_2(\mu) = \overline{x}_\mu$.

We deduce from the above observations, that any solution g to (55) entering in the domain $\{(x,y) \in \mathcal{D} \,|\, l_1(y) < x < l_2(y)\}$ remains in this domain. Since l_2 is bounded above by $\overline{x}_\mu$ on $[-\mu,\mu]$, it follows also that any solution g to (55) defined on a maximal interval I and passing through $(x_0,y_0) \in \{(x,y) \in \mathcal{D} \,|\, x \geq \overline{x}_\mu\}$ is strictly increasing and satisfies $g(y) > l_2(y)$ for all $y \in I$.

Now, let $(y_n)_{n\geq 0}$ an increasing sequence converging to μ. For each $n \in \mathbb{N}$ there exists a unique solution $g_{\overline{x}_\mu,y_n}$ to (55) satisfying $g_n(\overline{x}_\mu) = y_n$. Let us consider the sequence of functions $(g_n)_{n\geq 0}$ defined by the relations

$$\begin{cases} g_n(y) = g_{\overline{x}_\mu,y_n}(y) & \forall y \in (0,y_n], \\ g_n(y) = \overline{x}_\mu & \forall y \in [y_n,\mu]. \end{cases}$$

Our previous remarks on the solutions to (55) together with a standard non crossing property yield that, $(g_n)_{n\geq 0}$ is a decreasing sequence of increasing functions defined on $(0,\mu]$ and bounded above by $\overline{x}_\mu$. Thus, it admits a pointwise limit g defined on $(0,\mu]$. The function g is bounded above by $\overline{x}_\mu$ and satisfies $g(\mu) = \overline{x}_\mu$. We show below that g satisfies (55).

By construction, for each $n \in \mathbb{N}$, for any $y \in (0,\mu)$ one has

$$g_n(y) = \overline{x}_\mu - \int_y^\mu f(g_n(s),s)\,\mathbb{1}_{s\leq y_n}\,ds.$$

A direct computation shows that, for any fixed $y > 0$, the mapping $x \longrightarrow f(x,y)$ is continuously increasing over $\{x \,|\, x \geq l_2(y)\}$. We deduce that, for any $y \in (0,\mu)$,

$$\int_y^\mu \lim_{n\longrightarrow\infty} f(g_n(s),s)\,\mathbb{1}_{s\leq y_n}\,ds = \int_y^\mu f(g(s),s)\,ds \leq \int_y^\mu f(\overline{x}_\mu,s)\,ds < \infty,$$

[32]Note that, the definition of y^* in (33) implies that $\mu - r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu} = \mu(1 - r\sigma^2\frac{1}{\mu^2+2r\sigma^2}) > 0$.

where the last inequality comes from the fact that the mapping $s \longrightarrow f(\overline{x}_\mu, s)$ is continuous over $(0,\mu)$ with $\lim_{s\longrightarrow\mu} f(\overline{x}_\mu, s) = \frac{1}{2}\frac{1-r\sigma^2(\frac{y^*}{\mu})^2\frac{1}{\mu^2}}{r(\frac{y^*}{\mu})^2} < \infty$. It results from the dominated convergence Theorem that,

$$g(y) = \overline{x}_\mu - \int_y^\mu f(g(s), s)\, ds. \tag{120}$$

Thus, g is defined and increasing on $(0,\mu]$, satisfies the ode (55)-(56). A standard extension argument ensures that g is defined on a maximal interval $I \subset (-\mu,\mu)$ as well.

We show that $\overline{y}^{**} \equiv g^{-1}(0)$ is well defined and satisfies $\overline{y}^{**} > y^*$. Take the solution g_{0,y^*} to (55) defined on a maximal interval $I \subset (-\mu,\mu)$ passing through $(0, y^*)$. A computation shows that the function $v_1(y) = \frac{y^*}{\mu}(\phi(y^*) - \phi(y))$ defined on $(-\mu,\mu)$ satisfies $v_1(y^*) = g_{0,y^*}(y^*) = 0$ together with the inequality $v_1'(y) < f(v_1(y), y)$ for any $y \in (-\mu, 0]$. We deduce that $g_{0,y^*}(y) > v_1(y)$ for all $y \in (y^*, 0]$. From the Cauchy-Lipschitz Theorem it follows that $g_{0,y^*} > g_{v_1(0),0}$ on a maximal interval, where $g_{v_1(0),0}$ is the solution to (55) passing through $(v_1(0), 0)$.

Now, let us consider the function $v_2(y) = \frac{y^*}{\mu}(\phi(y\frac{y^*}{\mu}) + \phi(y^*)) \geq l_2(y)$ on $[0,\mu]$. Computations shows that $v_2(0) = v_1(0)$, $v_2(\mu) = \overline{x}_\mu$ and $v_2'(y) \leq f(v_2(y), y)$ for any $y \in [0,\mu]$. We deduce that $g_{0,y^*}(y) \geq v_2(y)$ for any $y \in [0,\mu]$. It follows that $g_{0,y^*} > g_{v_1(0),0} \geq g$ which implies that $\overline{y}^{**} \equiv g^{-1}(0) > y^*$.

Finally, we show that g is the unique solution to $g'(y) = f(g(y), y)$ satisfying the boundary condition $g(\mu) = \overline{x}_\mu$. Suppose the contrary, let g and $\tilde{g}$ be two solutions to (55) with $g(\mu) = \tilde{g}(\mu) = \overline{x}_\mu$ and $\tilde{g}(y) > \tilde{g}(y)$ over $(0,\mu)$. The functions g and $\tilde{g}$ satisfy (120). It follows that,

$$\tilde{g}(y) - g(y) \; = \; \int_y^\mu f(g(s), s) - f(\tilde{g}(s), s)\, ds. \tag{121}$$

The right hand side of (121) is strictly positive whereas its left hand side is negative because the mapping $x \longrightarrow f(x,y)$ is increasing for any fixed $y > 0$, thus, a contradiction.

We now turn to the study of the function $\overline{k} = (\phi - \overline{b})^{-1} : [\phi(\overline{y}^{**}), \infty) \longrightarrow [\overline{y}^{**}, \mu)$. We observe that

$$
\begin{aligned}
\phi'(y) - g'(y) &= \frac{\sigma^2}{\mu^2 - y^2}\left(1 - \frac{yy^* - \left(\mu - r\sigma^2\left(\frac{y^*}{\mu}\right)^2\frac{1}{\mu}\right)\psi\left(\frac{\mu}{y^*}\,g(y)\right)}{yy^* \;-\; \mu\psi\left(\frac{\mu}{y^*}\,g(y)\right)}\right) \\
&= \frac{\sigma^2}{\mu^2 - y^2}\,\frac{\frac{1}{\mu}r\sigma^2(\frac{y^*}{\mu})^2\psi\left(\frac{\mu}{y^*}\,g(y)\right)}{-yy^* + \mu\psi\left(\frac{\mu}{y^*}\,x\right)}
\end{aligned}
$$

is positive for $g(y) > \max(0, \frac{y^*}{\mu}\phi(g(y)\frac{y^*}{\mu})$, so that $\phi - \overline{b}$ is strictly increasing over $[\overline{y}^{**}, \mu)$ where $\overline{b} = \max(0, g)$ with g satisfying (55) and (57). Thus, $\overline{k} = (\phi - \overline{b})^{-1} : [\phi(\overline{y}^{**}), \infty) \longrightarrow [\overline{y}^{**}, \mu)$ is well defined, increasing and satisfies

$$
\begin{aligned}
\overline{k}'(z) &= \frac{1}{(\phi - \overline{b})'(\overline{k}(z))} \\
&= \frac{\mu^3}{y^* r\sigma^2}\,\frac{\mu^2 - k(z)^2}{\sigma^2}\,\frac{\psi\left(\frac{\mu}{y^*}(\phi(\overline{k}(z)) - z)\right)\mu - yy^*}{y^*\psi\left(\frac{\mu}{y^*}(\phi(\overline{k}(z)) - z)\right)}.
\end{aligned}
$$

Because $\overline{b} > 0$ and $\overline{k} = (\phi - \overline{b})^{-1}$ we have that $\phi(\overline{k}(z)) - z > 0$. Because $\overline{b}(y) > \frac{y^*}{\mu}\phi(\overline{b}(y)\frac{y^*}{\mu}$ we have that $\psi\left(\frac{\mu}{y^*}(\phi(\overline{k}(z)) - z)\right)\mu - yy^* > 0$, so that $(z, \overline{k}(z)) \in \widetilde{\mathcal{D}} = \{(z, y) \in \mathbb{R} \times (-\mu, \mu) \,|\, \phi(y) - z > \max(0, \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu}))\}$. Finally, the relation $\overline{k} = (\phi - \overline{b})^{-1}$ together with $\lim_{y \longrightarrow \mu} \overline{b}(y) = \overline{b}(\mu) = \overline{x}_\mu < \infty$ implies that $\lim_{y \longrightarrow \mu} \overline{k}^{-1}(y) = \infty$ and in turn that $\lim_{z \longrightarrow \infty} \overline{k}(z) = \infty$ since $\overline{k}$ is increasing. Then, posing $y = \overline{k}(z)$, the relation $\overline{b}(y) = \phi(y) - \overline{k}^{-1}(y)$ leads to $\lim_{z \longrightarrow \infty} \overline{b}(k(z)) = \phi(\overline{k}(z)) - z = \overline{x}_\mu$.

Conversely, let us consider a C^1-function k solution to the ordinary differential equation (58), (59) on the domain $\widetilde{\mathcal{D}}$. The function k is increasing so that, $b = \phi - k^{-1}$ is well defined over $(\overline{y}^{**}, \mu)$. It is positive since by assumption $(z, k(z)) \in \widetilde{\mathcal{D}}$. A direct computation shows that b satisfies the ordinary differential equation (58), (59) on the domain $\mathcal{D}$. Thus, we obtain that $b = \overline{b}$ and in turn $k = \overline{k}$. The proof of Lemma 3 is complete. □

Proof of Lemma 5. Since for any y fixed in $(-\mu, \mu)$, the mapping $x \longrightarrow \overline{V}_x(x, y)$ is concave on $[0, \infty)$, we only have to check that $\overline{V}_x(0, y) < \overline{p}$ for any $y \in (-\mu, \mu)$. Noting that $\overline{V}'_\mu(0) = \overline{p}$ and that $\lim_{y \longrightarrow \mu} \overline{V}_x(0, y) = \overline{V}'_\mu(0)$, the result follows from the fact that the mapping $y \longrightarrow \overline{V}_x(0, y)$ is increasing.[33] To see that latter point, note that $\overline{V}(x, y) = U(\phi(y) - x, y)$ where the function U is defined in the proof of Proposition 6.

[33]A computation based on (76) and (78) yields $\lim_{y \longrightarrow \mu} \overline{V}_x(0, y) = \overline{V}'_\mu(0)$.

We obtain that $\overline{V}_x(0,y) = -U_z(0,\phi(y))$ and thus that $\frac{d}{dy}\overline{V}_x(0,y) = -\phi'(y)U_{zz}(0,\phi(y))$. The result follows since ϕ is increasing and $U_{zz}(z,y) = \frac{y^*}{2\mu^2}\overline{k}'(z)(h_2''(\overline{k}(z))h_1(y) - h_1''(\overline{k}(z))h_2(y))$ which we know to be negative from the proof of the concavity of the mapping $x \longrightarrow \overline{V}(x,y)$ in Proposition 7. □

Proof of Proposition 9.

Proof of Assertion (i). From (44), (45) and (46) the mapping

$$y \longrightarrow G(z,y) = -h_1'(y)h_2(\psi(z)) + h_2'(y)h_1(\psi(z)) + \frac{2\mu^2}{y^*}p$$

is well defined over $[\psi(z),\infty)$, increasing and satisfies the equality $G(\psi(z),z) = \frac{2\mu^2}{y^*}(p-1) < 0$ and $\lim\limits_{y\longrightarrow\mu} G(z,y) = +\infty$. Thus, the function k is well defined over $[z_i,\infty)$ and satisfies the inequality $k > \psi$. Applying the implicit function Theorem, we deduce from the relation $G(z,k(z)) = 0$ that, the function k is differentiable and satisfies for any $z > z_i$

$$k'(z) = \psi'(z)\frac{h_2'(k(z))h_1'(\psi(z)) - h_1'(k(z))h_2'(\psi(z))}{h_1''(k(z))h_2(\psi(z)) - h_2''(k(z)h_1(\psi(z))}. \tag{122}$$

We saw in the proof of Proposition 7 that the denominator of the right hand side of (122) is negative. The numerator of k' in (122) is also negative. To see this point, remark that $x \longrightarrow h_2'(x)h_1'(y) - h_1'(x)h_2'(y)$ is decreasing over $[y,\mu)$ since it takes the value 0 at $x = y$, and its derivative has the same sign as $h_2(x)h_1'(y) - h_1(x)h_2'(y) < 0$. Thus, k is increasing over $[z_i,\infty)$. Then, assertion (i) of Propostion 9 follows from the next lemma.

Lemma 6 *the following holds*

(i) $\psi(z + x_\mu) < k(z)\ \ \forall z$,

(ii) $\psi(z + x_\mu + \epsilon) > k(z)\ \ \forall\epsilon > 0$ *for z sufficiently large.*

Proof of Lemma 6. We show that, for any z, $G(z,\psi(x_\mu + z)) < 0$ and that, for any $\epsilon > 0$ and any z sufficiently large, $G(z,\psi(x_\mu + z + \epsilon)) > 0$. Using (44), (45), (48), computations yield, for $x \geq 0$

$$G(z,\psi(x+z)) = g_1(x) + e^{-\beta z}g_2(x),$$

with

$$
\begin{aligned}
g_1(x) &= (-\frac{\mu^2}{y^*}+\mu)e^{\beta\gamma x} - (\frac{\mu^2}{y^*}+\mu)e^{-(\gamma-1)\beta x} + p\frac{2\mu^2}{y^*}, \\
&= 2(1-\gamma)\mu e^{\beta\gamma x} - 2\gamma\mu e^{-(\gamma-1)\beta x} + p\frac{2\mu^2}{y^*}. \\
g_2(x) &= (-\frac{\mu^2}{y^*}+\mu)e^{-\beta\gamma x} - (\frac{\mu^2}{y^*}+\mu)e^{(\gamma-1)\beta x} + p\frac{2\mu^2}{y^*} \\
&= 2(1-\gamma)\mu e^{-\beta\gamma x} - 2\gamma\mu e^{(\gamma-1)\beta x} + p\frac{2\mu^2}{y^*}. \qquad (123)
\end{aligned}
$$

To prove (i) and (ii), we show that the functions g_1 and g_2 are increasing and satisfiy $g_2(x_\mu) < g_1(x_\mu) = 0$. A computation leads to

$$g_1'(x) > 0 \Leftrightarrow e^{(1-2\gamma)\beta x} > 1, \text{ and } g_2'(x) > 0 \Leftrightarrow e^{-(1-2\gamma)\beta x} < 1.$$

Both inequalities hold true since $(1-2\gamma)\beta > 0$. The relation $g_1(x_\mu) = 0$ follows from the definition of x_μ. Finally, using (123) and (41) and rearrangging terms yield that

$$g_2(x_\mu) < 0 \Leftrightarrow \overline{g}(x_\mu) > \underline{g}(x_\mu) \qquad (124)$$

with $\overline{g}(x) = (1-\gamma)(e^{\gamma\beta x} - e^{-\beta\gamma x})$ and $\underline{g}(x) = \gamma(e^{(1-\gamma)\beta x} - e^{(\gamma-1)\beta x})$. It is easy to see that $\overline{g}(0) = \underline{g}(0) = 0$ and that $\overline{g}$ and $\underline{g}$ are decreasing. To prove (124), we remark that, for any $x > 0$, $\overline{g}'(x) > \underline{g}'(x)$. Indeed, this latter inequality is equivalent to

$$e^{\gamma\beta x} + e^{-\beta\gamma x} < e^{\beta(\gamma-1)x} + e^{\beta(1-\gamma)x},$$

which, given that $-\beta\gamma < \beta(1-\gamma)$, follows the properties of the function cosh. $\square$

Thus, from (i) and (ii) we have that,

$$\forall\epsilon > 0,\ \exists\overline{z},\ \forall z \geq \overline{z},\ \ \psi(z + x_\mu) < k(z) < \psi(z + x_\mu + \epsilon).$$

Assertions (95) and (96) follow by noting that, $k(z) = \psi((\phi(k(z)) - z) + z)$ and that ψ is a bounded continuous and increasing function. The proof of assertion (i) is complete.

Proof of Assertion (ii). We start with the existence of a solution z_i^* to equation (97). Let us consider

$$f(z) = \frac{p}{\psi'(z)}h_1(\psi(z)) + \int_z^\infty h_1'(k(u))\,du. \qquad (125)$$

To begin, we show that $f(z_i^*) > 0$ and that $\lim_{z \longrightarrow \infty} f(z) < 0$. Since function h_1' is negative and increasing, we deduce from the inequality $\psi < k$ that

$$\int_{z_i^*}^{\infty} h_1'(\psi(u))\, du < \int_{z_i^*}^{\infty} h_1'(k(u))\, du. \tag{126}$$

Thus, to show that $f(z_i^*) > 0$, we show that

$$\frac{p}{\psi'(z_i^*)} h_1(\psi(z^*)) + \int_{z_i^*}^{\infty} h_1'(\psi(u))\, du > 0.$$

Computations are explicit and yield that (126) is equivalent to

$$p(1 + e^{-\beta z_i^*}) - \frac{\gamma}{\gamma - 1} e^{-\beta z_i^*} + \frac{1-\gamma}{\gamma} > 0. \tag{127}$$

An easy computation shows that the left hand side of (127) is equal to zero when $p = 1$. This implies that $f(z_i^*) > 0$ for $p > 1$. We already know that $k = \psi$ when $p = 1$. We thus obtained as a by product result that $z_i^* = z^*$ when $p = 1$.

We show that $\lim_{z \longrightarrow \infty} f(z) = 0^-$. From (48) and (96), it is sufficient to show that,

$$\frac{p}{\beta} e^{\beta\gamma z}(1 + e^{-\beta z}) + \int_{z}^{\infty} h_1'(\psi(u + x_\mu))\, du < 0$$
$$\Leftrightarrow p(1 + e^{-\beta z}) < \frac{1-\gamma}{-\gamma} e^{\beta\gamma x_\mu}.$$

This latter inequality follows from (43), thus the result. Therefore, there exists z_i^* such that $f(z_i^*) = 0$.

Uniqueness of z_i^*. A direct computation shows that $f'(z) < 0$ for $z < 0$. Therefore, given that $\lim_{z \longrightarrow \infty} f(z) < 0$, if f has more than one zero, there exists z_1 and z_2 such that $0 < z_1 < z_2$, $f(z_1) = f(z_2) = 0$ and $f'(z_1) > 0$ and $f'(z_2) < 0$. We reason by way of contradiction and prove that if there are z_1 and z_2 such that $0 < z_1 < z_2$, $f(z_1) = f(z_2) = 0$ and $f'(z_1) > 0$ then $f'(z_2) > 0$ which contradicts $\lim_{z \longrightarrow \infty} f(z) < 0$.

We consider below $g(z) = f(z) h_2(\psi(z))$ that as the same zeros and the same sign than f. From (46) and (47), we have that

$$g(z) = p\sigma^2 + \int_{z}^{\infty} h_1'(k(u))\, du\; h_2(\psi(u)).$$

Thus,

$$\begin{aligned} g'(z) &= \psi'(z) h_2'(\psi(z)) \int_{z}^{\infty} h_1'(k(u))\, du - h_2(\psi(z)) h_1'(k(z)) \\ &= \psi'(z) h_2'(\psi(z)) \int_{z}^{\infty} h_1'(k(u))\, du - p\frac{2\mu^2}{y^*} - h_1(\psi(z)) h_2'(k(z)), \end{aligned}$$

where the last equality follows from (94).

Now, z_1 satisfies by assumption $g(z_1) = 0$ and $g'(z_1) > 0$, thus

$$\begin{aligned} g'(z_1) &= \psi'(z_1)h_2'(\psi(z_1))(-\frac{p}{\psi'(z_1)}h_1(\psi(z_1)) - p\frac{2\mu^2}{y^*} - h_1(\psi(z_1))h_2'(k(z_1)) \\ &= -ph_2'(\psi(z_1))h_1(\psi(z_1)) - p\frac{2\mu^2}{y^*} - h_1(\psi(z_1))h_2'(k(z_1)) > 0, \end{aligned} \tag{128}$$

where the first term of the right hand side of (128) follows from (125). Any zero z of g satisfies

$$g'(z) = -ph_2'(\psi(z))h_1(\psi(z)) - p\frac{2\mu^2}{y^*} - h_1(\psi(z))h_2'(k(z)).$$

We show that

$$q_1 : \ z \longrightarrow -h_2'(\psi(z))h_1(\psi(z)),$$

and

$$q_2 : \ z \longrightarrow -h_1(\psi(z))h_2'(k(z))$$

are increasing functions which will imply that $g'(z_2) > 0$. We have

$$sign\{q_1'(z))\} = sign\{-2r\sigma^2 - (\frac{\mu^2}{y^*} - \psi(z))(-\frac{\mu^2}{y^*} - \psi(z))\},$$

and

$$-2r\sigma^2 - (\frac{\mu^2}{y^*} - \psi(z))(-\frac{\mu^2}{y^*} - \psi(z)) = \mu^2 - \psi^2(z) > 0.$$

Also, we have

$$q_2'(z) = -k'(z)h_2''(k(z))h_1(\psi(z)) - h_1'(\psi(z))\psi'(z)h_2'(k(z)).$$

It follows that

$$q_2'(z) \geq 0 \Leftrightarrow k'(z)\frac{2r\sigma^2}{\mu^2 - k(z)^2} < \psi'(z)\frac{h_1'(\psi(z))h_2(k(z))(-\frac{\mu^2}{y^*} - k(z))}{-h_2(k(z))h_1(\psi(z))}. \tag{129}$$

Using (122), a computation yields that

$$k'(z)\frac{2r\sigma^2}{\mu^2 - k(z)^2} = \psi'(z)\frac{h_1'(\psi(z))h_2(k(z))(-\frac{\mu^2}{y^*} - k(z)) - h_1(k(z))h_2'(\psi(z))(\frac{\mu^2}{y^*} - k(z))}{h_1(k(z))h_2(\psi(z)) - h_2(k(z))h_1(\psi(z)}. \tag{130}$$

Using (129) and (130), a computation shows that $q_2'(z) \geq 0$ is equivalent to $\psi(z) < k(z)$, which we know to be true, thus the result.

Proof of Assertion (iii). Given assertion (i), we only need to prove that the solution k_1 to the ordinary differential equation

$$\begin{aligned} k_1'(z) &= \Theta(z, k_1(z)), \\ k_1(z_i^*) &= k_2(z_i^*), \end{aligned}$$

with

$$\Theta(z,y) = \frac{\mu^3}{y^* r\sigma^2} \frac{\mu^2 - y^2}{\sigma^2} \frac{\psi\left(\frac{\mu}{y^*}(\phi(y) - z)\right)\mu - yy^*}{y^*\psi\left(\frac{\mu}{y^*}(\phi(y) - z)\right)}$$

defined on the domain $\widetilde{\mathcal{D}} = \{(z,y) \in \mathbb{R} \times (-\mu, \mu) \,|\, \phi(y) - z > \max(0, \frac{y^*}{\mu}\phi(y\frac{y^*}{\mu}))\}$, is a well defined continuously differentiable and increasing function over $(-\infty, z_i^*)$.

We deduce from the proof of Lemma 3 that the ode

$$\begin{aligned} k'(z) &= \Theta(z, k(z)), \\ k(z_i) &= y_i, \end{aligned}$$

where the couple (z_i, y_i) satisfies

$$\phi(y_i) - z_i > \max(0, \frac{y^*}{\mu}\phi(y_i\frac{y^*}{\mu})) \tag{131}$$

has a unique solution that is continuously differentiable and increasing over $(-\infty, z_i)$. To establish Assertion (iii), it thus remain to show that the couple $(z_i^*, k_2(z_i^*))$ satisfies (131). We deduce from (74) and (75) that this requirement is equivalent to

$$h_1'(k_2(z_i^*))h_2(\psi(z_i^*)) + h_2'(k_2(z_i^*))h_1(\psi(z_i^*)) > 0. \tag{132}$$

Consider first the case $z_i^* \leq 0$. Let us recall that the mapping $y \longrightarrow h_1'(y)h_2(\psi(z)) + h_2'(y)h_1(\psi(z))$ is increasing and that $k_2 > \psi$. It follows that

$$\begin{aligned} & h_1'(k_2(z_i^*))h_2(\psi(z_i^*)) + h_2'(k_2(z_i^*))h_1(\psi(z_i^*)) \\ > \; & h_1'(\psi(z_i^*))h_2(\psi(z_i^*)) + h_2'(\psi(z_i^*))h_1(\psi(z_i^*)) = -2\psi(z_i^*) \geq 0. \end{aligned}$$

Thus, (132) is satisfied. Next, consider that $z_i^* > 0$. The function k_2 satisfies (94). It follows that (132) is equivalent to $h_2'(k_2(z_i^*))h_1(\psi(z_i^*)) + \frac{\mu^2}{y^*}p > 0$. From the proof of assertion (ii) we know that z_i^* satisfies[34]

$$-ph_2'(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{2\mu^2}{y^*} - h_1(\psi(z_i^*))h_2'(k(z_i^*)) < 0.$$

[34] See the argument underlying Equation (128).

that is

$$p\frac{\mu^2}{y^*} + h_1(\psi(z_i^*))h_2'(k(z_i^*)) > -ph_2'(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{\mu^2}{y^*}.$$

Then, a direct computation shows that

$$-ph_2'(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{\mu^2}{y^*} > 0$$

if and only if $\psi(z_i^*) > 0$ which is true since we consider $z_i^* > 0$. As a final remark, an easy computation based on (74) and (122) yields that $k_1'(z) \neq k_2'(z)$ so that k^* is not differentiable at $\{z_i^*\}$. The proof of assertion (iii) is complete. □

The next Lemma completes the proof of Proposition 13.

Lemma 7 *The following holds*

$$\frac{k^{*'}(z)}{\psi'(z+\tilde{b}(z))} - 1 > 0 \ \ \textit{for any } z > z_i^*.$$

Proof of Lemma 7. From Proposition 9 and Proposition 12, we deduce that k^* satisfies (122). Therefore, to prove Lemma 7, we show that

$$\frac{\psi'(z)}{\psi'(z+x)}\frac{h_2'(\psi(z+x))h_1'(\psi(z)) - h_1'(\psi(z+x))h_2'(\psi(z))}{h_1''(\psi(z+x))h_2(\psi(z)) - h_2''(\psi(z+x))h_1(\psi(z))} - 1 < 0.$$

To prove the latter inequality, it is enough to prove

$$\begin{aligned}&\psi'(z)(h_2'(\psi(z+x))h_1'(\psi(z)) - h_1'(\psi(z+x))h_2'(\psi(z)))\\ -&\psi'(z+x)(h_1''(\psi(z+x)h_2(\psi(z)) - h_2''(\psi(z+x))h_1(\psi(z))) > 0\end{aligned}$$

or equivalently,

$$\begin{aligned}&\psi'(z)(h_1(\psi(z+x))h_2(\psi(z))(-\frac{\mu^2}{y^*} - \psi(x+z))(\frac{\mu^2}{y^*} - \psi(z))\\ &-h_2(\psi(z+x))h_1(\psi(z))(\frac{\mu^2}{y^*} - \psi(x+z))(-\frac{\mu^2}{y^*} - \psi(z)))\\ -&\frac{\psi'(z+x)2\sigma^2 r}{(\mu^2-\psi^2(x+z))^2}(h_1(\psi(z+x))h_2(\psi(z)) - h_2(\psi(z+x))h_1(\psi(z)) > 0.\end{aligned}$$

This latter expression is equivalent to

$$\begin{aligned}&h_1(\psi(z+x))h_2(\psi(z))\left(\psi'(z)(-\frac{\mu^2}{y^*} - \psi(x+z))(\frac{\mu^2}{y^*} - \psi(z)) - \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\right) +\\ &h_2(\psi(z+x))h_1(\psi(z))\left(\psi'(z)(-\frac{\mu^2}{y^*} + \psi(x+z))(-\frac{\mu^2}{y^*} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\right) > 0.\end{aligned} \tag{133}$$

We remark that

$$\left(\psi'(z)(-\frac{\mu^2}{y^*}+\psi(x+z))(-\frac{\mu^2}{y^*}-\psi(z))+\frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\right)>0, \tag{134}$$

and

$$h_2(\psi(z+x))h_1(\psi(z))>h_1(\psi(z+x))h_2(\psi(z)). \tag{135}$$

A straightforward computation shows that the sign of

$$\begin{aligned}&\psi'(z)(-\frac{\mu^2}{y^*}+\psi(x+z))(-\frac{\mu^2}{y^*}-\psi(z))+\frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\\&+\psi'(z)(-\frac{\mu^2}{y^*}-\psi(x+z))(\frac{\mu^2}{y^*}-\psi(z))-\frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\end{aligned}$$

is the same as the sign of $-\psi(z)+\psi(x+z)$. But clearly, $-\psi(z)+\psi(x+z)$ is positive. It then follows from (133), (134) and (135) that the mapping $y \longrightarrow b^*(y)$ is decreasing over $[y^{i*},\mu)$.